1. p. 301, #2. The number of tails is negative binomial with parameters $r = 5$ and $p = 1/30$.

(a) 

$$E(X) = \frac{r(1-p)}{p} = \frac{5 \times 29}{1} = 145.$$ 

(b) 

$$\text{Var}(X) = \frac{r(1-p)}{p^2} = 4350.$$ 

2. p. 301, #3.

(a) Let $X$ denote the number of heads obtained before 5 heads are obtained and let $Y$ denote the total number of tosses that are required. Then $Y = X + 5$ and $E(Y) = E(X) + 5 \Rightarrow E(Y) = 150$ (using #2).

(b) Since $Y = X + 5$, $\text{Var}(Y) = \text{Var}(X) = 4350$.

3. p. 345, #5. Note that $\sum X_i \sim \text{Pois}(4\lambda)$. The event of interest is $\{\bar{X} < .5\} = \{\sum X_i < 2\}$; hence,

$$\Pr\left(\sum X_i < 2\right) = \Pr\left(\sum X_i = 0\right) + \Pr\left(\sum X_i = 1\right)$$

$$= \exp(-4\lambda) + 4\lambda \exp(-4\lambda)$$

$$= (1 + 4\lambda) \exp(-4\lambda).$$

4. p. 345, #9. The total number of people that arrive in one minute is the sum of the number of men and the number of women, and this sum is Poisson in distribution with mean 3. Using R, the call `ppois(4, lambda = 3)` computes the probability of 4 or less arrivals. The numerical value is .8152.

5. p. 345, #11. The probability that at least one child will be successful on a given Sunday is $1/3 + 1/5 - 1/15 = 7/15$. The number of Sundays until at least one of the two children has a successful launch is geometric in distribution with $p = 7/15$, and the expected number of Sundays until a successful launch is

$$\frac{1-p}{p} = \frac{8}{7}.$$ 

6. p. 287, #3. Let $X$ denote the number of red balls. Then, the expected number of red items is

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \frac{nA}{A+B} = \frac{7}{3},$$

and

$$\text{Var}(X) = \frac{nAB}{(A+B)^2} \frac{A+B-n}{A+B-1} = \frac{8}{9}.$$ 

Since $n = 7$, $\bar{X} = X/n$, we have $E(\bar{X}) = 1/3$ and $\text{Var}(\bar{X}) = \text{Var}(X)/n^2 = 8/411.$
7. p. 280, #11. Note that the lower limit on the sum can be changed to 0 without affecting the sum. Then,
\[ \sum_{x=2}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} \]
\[ = E[X(X-1)] \]
\[ = E(X^2) - E(X) \]
\[ = \text{Var}(X) + [E(X)]^2 - E(X) \]
\[ = np(1-p) + (np)^2 - np = n(n-1)p^2. \]

8. p. 272, #15. Let \( v = \text{Var}(X_1 + \cdots + X_n) = \sum_i \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j). \) Then,
\[ 0 \leq v = n\sigma^2 + n(n-1)\rho \sigma \]
\[ \Rightarrow \rho \geq -1/(n-1). \]

\[ E[(X - \mu)^3] = E(X^3) - 3E(X^2)\mu + 3E(X)\mu^2 - \mu^3 \]
\[ = E(X^3) - 3\mu(\mu^2 + \sigma^2) + 3\mu^3 - \mu^3 \]
\[ = E(X^3) - 3\mu \sigma^2 - \mu^3. \]

10. p. 272, #19.
\[ c'(t) = \frac{\psi'(t)}{\psi(t)} \]
\[ c''(t) = \frac{\psi(t)\psi''(t) - [\psi'(t)]^2}{[\psi(t)]^2} \]

Since \( \psi(0) = 1, \psi'(0) = \mu, \) and \( \psi''(0) = E(X^2) = \mu^2 + \sigma^2, c'(0) = \mu \) and \( c''(0) = \sigma^2. \)


(a) The marginal p.d.f. of \( X \) is
\[ f_1(x) = \int_0^x 8xy \, dy = 4x^3, \text{ for } 0 < x < 1. \]

Therefore, the conditional p.d.f. of \( Y \) given that \( X = .2 \) is
\[ g_1(y|X = .2) = \frac{f(2, y)}{f_1(2)} = 50y, \text{ for } 0 < y < .2. \]

The conditional mean minimizes the mean square error, and it is
\[ E(Y|X = .2) = \frac{2}{15} = .1333. \]
(b) The median of $g_1(y|X = .2)$ is $m = \left(\frac{1}{30}\right)^5 = .1414$ ($m$ minimizes the M.A.E.)

12. p. 264 #7. The marginal p.d.f. of $X$ is

$$f_1(x) = \int_0^1 (x + y) \, dy = x + \frac{1}{2}, \text{ for } 0 \leq x \leq 1.$$ 

Therefore, for $0 \leq x \leq 1$, the conditional p.d.f. of $Y$ given that $X = x$ is

$$g_2(y|x) = \frac{f(x, y)}{f_1(x)} = \frac{2(x + y)}{2x + 1}, \text{ for } 0 \leq y \leq 1.$$ 

Hence,

$$E(Y|X) = \int_0^1 \frac{2(xy + y^2)}{2x + 1} \, dy = \frac{3x + 2}{3(2x + 1)}.$$ 

and

$$E(Y^2|X) = \int_0^1 \frac{2(xy^2 + y^3)}{2x + 1} \, dy = \frac{4x + 3}{6(2x + 1)}.$$ 

and

$$\text{Var}(Y|X) = \frac{4x + 3}{6(2x + 1)} - \left[ \frac{3x + 2}{3(2x + 1)} \right]^2 = \frac{1}{36} \left[ 3 - \frac{1}{(2x + 1)^2} \right].$$


(a) The best predictor is the mean

$$E(Y) = \int_0^1 \int_0^1 y \cdot \frac{2}{5}(2x + 3y) \, dx \, dy = \frac{3}{5}.$$ 

(b) The best prediction is the median $m$ of $X$. The marginal p.d.f. of $X$ is

$$f_1(x) = \int_0^1 \frac{2}{5}(2x + 3y) \, dy = \frac{4x + 3}{5} \text{ for } 0 \leq x \leq 1.$$ 

$m$ satisfies

$$\frac{1}{2} = \int_0^m \frac{4x + 3}{5} \, dx.$$ 

Hence, $4m^2 + 6m - 5 = 0 \Rightarrow m = \frac{\sqrt{29} - 3}{4}$. 

14. p. 255. # 3. Since the p.d.f. of $X$ is symmetric with respect to 0, $E(X) = 0$ and $E(X^k) = 0$ for every odd positive integer. Thus, $E(XY) = E(X^7) = 0$. Since $E(XY) = 0$ and $E(XY) = E(X)E(Y) = 0$, Cov$(X, Y) = 0$.

15. p. 255. # 18. Since $f(x, y) = f(y, x)$, $E(X) = E(Y)$, and

$$E(X) = \int_0^1 \int_0^1 x(x + y) \, dx \, dy$$

$$+ = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$  

Also,

$$E(XY) = \int_0^1 \int_0^1 xy(x + y) \, dx \, dy$$

$$= \int_0^1 \left( \frac{y}{3} + \frac{y^2}{2} \right) \, dy$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$  

Then,

$$\text{Cov}(X, Y) = \frac{1}{3} - \left( \frac{7}{12} \right) = -0.0095.$$

16. p. 240 #6. Since $f(x|a, b) = (b-a)^{-1} I_{(a,b)}(x)$,

$$\psi(t) = \int_a^b \frac{e^{tx}}{b-a} \, dx = \frac{e^{tb} - e^{ta}}{t(b-a)}.$$  

17. p. 233 #1. First, $E(X) = \frac{1}{2}$ since the distribution of $X$ is symmetric about $\frac{1}{2}$. Then,

$$E(X^2) = \int_0^1 x^2 \, dx = \frac{1}{3}.$$  

Then, $\text{Var}(X) = \frac{1}{3} - \left( \frac{1}{2} \right)^2 = \frac{1}{12}.$

18. p. 233 #5. Let $\mu = E(X)$ and $\sigma^2 = \text{Var}(X)$, and note that $E(X^2) = \sigma^2 + \mu^2$.

$$E[(X - \mu)^2] = E(X^2) - 2\mu \mu + \mu^2$$

$$= \sigma^2 + \mu^2 - 2\mu \mu + \mu^2$$

$$= \sigma^2 + (\mu - \mu).$$  

19. p. 233 #11. The c.d.f. of $X \sim \text{Unif}(a, b)$ is

$$F(x) = \begin{cases} 
0, & x < a \\
\frac{x - a}{b - a}, & a \leq x \leq b \\
1, & b < x.
\end{cases}$$
and so the quantile function is $x = F^{-1}(p)$ after restricting $F$ to $(a, b)$. In this case of $X \sim \text{Unif}(0, 1)$, $F^{-1}(p) = x$. Then

$$
\text{IQR} = F^{-1}(0.75) - F^{-1}(0.25) = 0.5.
$$