Page 129 Exercise 5: Suppose that the joint p.d.f. of two random variables $X$ and $Y$ is as follows:

$$f(x,y) = \begin{cases} 
  c(x^2 + y) & \text{for } 0 \leq y \leq 1 - x^2, \quad -1 < x < 1 \\
  0 & \text{otherwise}
\end{cases}$$

Determine (a) the value of the constant $c$; (b) $\Pr(0 \leq X \leq \frac{1}{2})$; (c) $\Pr(Y \leq X + 1)$; (d) $\Pr(Y = X^2)$.

Solution: We must integrate over the support, $0 \leq y \leq 1 - x^2$. Thus

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx \, dy = \int_{-1}^{1} \int_{0}^{1-x^2} c(x^2 + y) \, dy \, dx = \int_{-1}^{1} \int_{y=0}^{1-x^2} c(x^2 + y) \, dy \, dx$$

But,

$$c \int_{-1}^{1} \left( y^2 + \frac{1}{2} y^2 \right) \left|_{0}^{1-x^2} \right. \, dx = c \int_{-1}^{1} \left( (1 - x^2)x^2 + \frac{1}{2}(1 - x^2)^2 \right) \, dx = c \int_{-1}^{1} x^2 - x^4 + \frac{1}{2} - x^2 + \frac{x^2}{2} \, dx$$

$$= c \left( \frac{1}{2} x - \frac{1}{10} \right) \bigg|_{-1}^{1} = c \left( \frac{1}{2} - \frac{1}{10} - \left( -\frac{1}{2} - \frac{-1}{10} \right) \right)$$

$$= c \left( \frac{1}{2} + \frac{1}{2} - \frac{1}{10} - \frac{-1}{10} \right) = c \left( 1 - \frac{2}{10} \right) = \frac{4}{5}.$$ 

Thus $c = \frac{5}{4}$.

Solution(b): Now the support has changed. Thus, along similar lines as in part(a),

$$\Pr \left( 0 \leq X \leq \frac{1}{2} \right) = \int_{0}^{1/2} \int_{0}^{1-x^2} \frac{5}{4} (x^2 + y) \, dy \, dx = \frac{79}{256}.$$ 

Solution(c): Here also support has changed. Thus

$$\Pr \left( Y \leq X + 1 \right) = 1 - \int_{-1}^{0} \int_{x+1}^{1-x^2} \frac{5}{4} (x^2 + y) \, dy \, dx = \frac{13}{16}.$$ 

Solution(d): The probability the $(X,Y)$ will lie on the curve $y = x^2$ is 0 for every continuous joint distribution.

Page 129 Exercise 6: Suppose that a point $(X,Y)$ is chosen at random from the region $S$ in the $xy$-plane containing all points $(x,y)$ such that $x \geq 0, y \geq 0$, and $4y + x \leq 4$.

a. Determine the joint p.d.f. of $X$ and $Y$.

b. Suppose that $S_0$ is a subset of the region $S$ having area $\alpha$ and determine $\Pr[(X,Y) \in S_0]$.

Solution(a): First, we find the area of the triangle that defines the support: $\frac{1}{2}(\text{Base}) \cdot (\text{Height}) = \frac{1}{2} \cdot 4 \cdot 1 = 2$. Thus since the area of $S$ is 2, and the joint p.d.f. is to be constant over $S$, then the value of the normalizing constant must be $\frac{1}{2}$.

Solution(b): The probability that $(X,Y)$ will belong to any subset $S_0$ is proportional to the area of that subset. Therefore,

$$\Pr[(X,Y) \in S_0] = \int \int_{S_0} \frac{1}{2} \, dx \, dy = \frac{1}{2} (\text{area of } S_0) = \frac{\alpha}{2}.$$ 

Page 129 Exercise 8: Suppose that $X$ and $Y$ are random variables such that $(X,Y)$ must belong to the rectangle in the $xy$-plane containing all points $(x,y)$ for which $0 \leq x \leq 3$ and $0 \leq y \leq 4$. Suppose also that the joint c.d.f. of $X$ and $Y$ at every point $(x,y)$ in this rectangle is specified as follows:

$$F(x,y) = \frac{1}{156} xy(x^2 + y).$$
a. Determine \( \Pr(1 \leq X \leq 2, 1 \leq Y \leq 2) \).
\[
= \Pr(X \leq 2, 1 \leq Y \leq 2) - \Pr(X \leq 1, 1 \leq Y \leq 2)
= [\Pr(X \leq 2, Y \leq 2) - \Pr(X \leq 2, Y \leq 1)] - [\Pr(X \leq 1, Y \leq 2) - \Pr(X \leq 1, Y \leq 1)]
= [F(2, 2) - F(2, 1)] - [F(1, 2) - F(1, 1)] = \frac{1}{156} ([24 - 10] - [6 - 2]) = \frac{1}{156}(10) = \frac{10}{156}
\]

b. Determine \( \Pr(2 \leq X \leq 4, 2 \leq Y \leq 4) \).
\[
= \Pr(2 \leq X \leq 3, 2 \leq Y \leq 4) \quad \text{(since \( \Pr(3 \leq X \leq 4) = 0 \))}
= \frac{1}{156} ([156 - 66] - [64 - 24]) = \frac{50}{156}
\]

c. Determine the cumulative distribution function of \( Y \).
\[
F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y) = F_X(3, y) = \frac{1}{156}xy(x^2 + y) \bigg|_{x=3} = \frac{1}{156}(3)y(3^2 + y) = \frac{52}{52}y(9 + y),
\]
so that
\[
F_Y(y) = \begin{cases} 
0 & \text{if } y < 0 \\
\frac{52}{52}y(9 + y) & \text{if } 0 \leq y \leq 4 \\
1 & \text{if } y > 1 
\end{cases}
\]

d. Determine the joint p.d.f of \( X \) and \( Y \).
\[
\text{For } 0 < x < 3 \text{ and } 0 < y < 4 \text{ (we exclude the boundary of the rectangular support set because these are sharp corner points and, hence, } F(x, Y) \text{ is not differentiable at those points).}
\]
\[
f_{X,Y}(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial y} \frac{1}{156}xy(x^2 + y) \right] = \frac{1}{156} \cdot \frac{\partial}{\partial x} [x^3 + 2xy]
= \frac{1}{156}(3x^2 + 2y).
\]
Then for \((x, y) \in \mathbb{R}^2 \setminus [0, 3] \times [0, 4] \)
\[
\frac{\partial^2 F(x, y)}{\partial x \partial y} = 0
\]
because \( F(x, y) = 0 \) or 1. At the boundaries we set \( f(x, y) = 0 \) arbitrarily. This doesn’t matter since the boundaries represent sets of probability zero. Therefore the joint p.d.f of \( X \) and \( Y \) is
\[
f(x, y) = \frac{1}{156}(3x^2 + 2y)I_{(0,3) \times (0,4)}(x, y)
\]
e. Determine \( \Pr(Y \leq X) \).
\[
\text{We need to find the probability of being in the region denoted by } A \text{ for this joint density, namely:
}\]
\[
\Pr(Y \leq X) = \int_0^3 \int_0^x \frac{1}{156}(3x^2 + 2y) dy dx
= \frac{1}{156} \int_0^3 (3x^2y + y^2) \bigg|_0^x dx
= \frac{1}{156} \int_0^3 (3x^3 + x^2) dx = \frac{1}{156} \left[ \frac{3x^4}{4} + \frac{x^3}{3} \right]_0^3
= \frac{1}{156} \left[ \frac{243}{4} + 9 \right] = \frac{279}{624} = \frac{93}{208} = 0.447.
\]
Exercise 10: Let $Y$ be the rate (calls per hour) at which calls arrive at a switchboard. Let $X$ be the number of calls during a 2-hour period. A popular choice of joint pdf for $(X, Y)$ in this example would be one like:

$$f_{X,Y}(x, y) = \begin{cases} \frac{(2y)^x}{x!} e^{-3y} & \text{if } y > 0 \text{ and } x = 0, 1, \ldots, \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that $f$ is a joint pdf. To verify that this is a valid probability function, we need to show that if we sum it over all values of $x$ and integrate it over the region $S_y = \{ y : y > 0 \}$, we get the value 1. Doing this:

$$\int_{0}^{\infty} \sum_{x=0}^{\infty} f_{X,Y}(x, y) dy = \int_{1}^{\infty} \sum_{x=0}^{\infty} \frac{(2y)^x}{x!} e^{-3y} dy = \int_{1}^{\infty} e^{-3y} \sum_{x=0}^{\infty} \frac{(2y)^x}{x!} dy$$

$$= \int_{1}^{\infty} e^{-3y} e^{2y} dy \quad \text{(recognizing the power series for } e^{2y})$$

$$= \int_{1}^{\infty} e^{-y} dy = -e^{-y} \bigg|_{1}^{\infty} = 0 - (-1) = 1, \text{ as required.}$$

b. Find $\Pr(X = 0)$.

$$\Pr(X = 0) = f_X(0) = \int_{0}^{\infty} f_{X,Y}(0, y) dy = \int_{0}^{\infty} \frac{(2y)^0}{0!} e^{-3y} dy = \int_{0}^{\infty} e^{-3y} dy = \frac{1}{3} e^{-3} \bigg|_{0}^{\infty} = 0 - (-1/3) = \frac{1}{3}.$$ 

Page 140 Exercise 2: Suppose that $X$ and $Y$ have a discrete joint distribution for which the joint p.f. is defined as follows:

$$f(x, y) = \begin{cases} \frac{1}{30}(x+y) & \text{for } x = 0, 1, 2 \text{ and } y = 0, 1, 2, 3, \\ 0 & \text{otherwise} \end{cases}$$

a. Determine the marginal p.f.’s of $X$ and $Y$.

The marginal p.f. for $X$ is:

$$\Pr(X = x) = f_X(x) = \sum_{y=0}^{3} f_{X,Y}(x, y) = \sum_{y=0}^{3} \frac{1}{30}(x+y)$$

$$\implies f_X(x) = \frac{1}{30} \sum_{y=0}^{3} (x+y) = \frac{1}{30} [(x+0) + (x+1) + (x+2) + (x+3)]$$

$$= \frac{1}{30} (4x+6) = \frac{1}{15}(2x+3), \ x = 0, 1, 2.$$

The marginal p.f. for $Y$ is:

$$\Pr(Y = y) = f_Y(y) = \sum_{x=0}^{2} f_{X,Y}(x, y) = \sum_{x=0}^{2} \frac{1}{30}(x+y)$$

$$\implies f_Y(y) = \frac{1}{30} \sum_{x=0}^{2} (x+y) = \frac{1}{30} [(0+y) + (1+y) + (2+y)]$$

$$= \frac{1}{30} (3y+3) = \frac{1}{10}(y+1), \ y = 0, 1, 2, 3.$$

b. Are $X$ and $Y$ independent?

For discrete random variables $X, Y$ to be independent, we must have:

$$\Pr(X = x, Y = y) = \Pr(X = x) \Pr(Y = y), \text{ for all } (x, y).$$
Let \( x = 1, \ y = 0 \). Computing: \( \Pr(X = 1) = 7/40, \ \Pr(Y = 0) = 6/40, \) and \( \Pr(X = 1, Y = 0) = 1/40 \). But \( \Pr(X = 1, Y = 0) = 1/40 \neq (7/40)(6/40) = \Pr(X = 1) \Pr(Y = 0) \), so \( X \) and \( Y \) are not independent.

**Page 140 Exercise 4:** Suppose that the joint p.d.f of \( X \) and \( Y \) is as follows:

\[
f(x, y) = \begin{cases} 
\left( \frac{15}{4} \right) x^2 & \text{for } 0 \leq y \leq 1 - x^2 \text{ and } -1 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

a. Determine the marginal p.d.f’s of \( X \) and \( Y \).

The marginal p.d.f for \( X \) is: \( f_X(x) = \int_0^{1-x^2} f_{X,Y}(x,y) dy = \int_0^{1-x^2} \left( \frac{15}{4} \right) x^2 dy \) for \(-1 \leq x \leq 1 \). Therefore,

\[
\Rightarrow f_X(x) = \left( \frac{15}{4} \right) x^2 1^{1-x^2} = \left( \frac{15}{4} \right) x^2 (1 - x^2).
\]

Since \( 0 \leq y \leq 1 - x^2 \), then \( x^2 \leq 1 - y \Rightarrow -\sqrt{1 - y} \leq x \leq \sqrt{1 - y} \), \( 0 \leq y \leq 1 \), then the marginal p.d.f for \( Y \) is: \( f_Y(y) = \int_{\sqrt{1-y}}^{\sqrt{1-y}} \left( \frac{15}{4} \right) x^2 dx \). That is

\[
f_Y(y) = \frac{5}{4} x^3 \bigg|_{\sqrt{1-y}}^{\sqrt{1-y}} = \frac{5}{4} \left( (1 - y)^{3/2} + (1 - y)^{3/2} \right) = \frac{5}{2} (1 - y)^{3/2}, \quad 0 \leq y \leq 1.
\]

b. Are \( X \) and \( Y \) independent?

Since the limits of integration of the joint p.d.f of \( X, Y \), namely \( 0 \leq y \leq 1 - x^2 \), cannot be stated separately for \( X \) and \( Y \), \( X \) and \( Y \) are not independent. Putting it another way, since the support of \( X \) & \( Y \) is not the cross product set of the support of \( X \) \((-1 \leq x \leq 1\) and the support of \( Y \) \(0 \leq y \leq 1\), then \( X \) and \( Y \) are not independent.

**Page 141 Exercise 6:** (a) Since \( X \) and \( Y \) are independent,

\[
f(x, y) = f_1(x) f_1(y) = g(x) g(y) = \frac{9}{64} x^2 y^2 I_{1}((r, s) : 0 \leq z \leq 2, 0 \leq y \leq 2).
\]

(b) Since \( X \) and \( Y \) have a continuous joint distribution \( \Pr(X = Y) = 0 \).
(c) Since \( X \) and \( Y \) are independent with the same probability distribution, \( \Pr(X > Y) = \Pr(Y < X) \). Since \( \Pr(X = Y) = 0 \), it follows that \( .5 = \Pr(X > Y) = \Pr(Y < X) \), and \( \Pr(X > Y) = .5 \).
(d) \( \Pr(X + Y \leq 1) = \Pr(X \leq 1 - Y) \) and

\[
\Pr(X + Y \leq 1) = \int_{0}^{1} \int_{0}^{1-y} f(x, y, ) \ dx \ dy = \frac{1}{1280}.
\]

**Page 141 Exercise 10:** (a) \( f(x, y) \) is constant over the circle \( S \). The area of \( S \) is \( \pi \) units, and so \( f(x, y) = 1/\pi \) for \((x, y) \in S \) and 0 everywhere else. The possible values of \( X \) are in the interval \([-1, 1]\), and given some value of \( x \in [-1, 1] \), \( -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2} \). Hence,

\[
f_1(x) = \frac{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{dy}{\pi}}{\pi} = \frac{2}{\pi} \sqrt{1 - x^2} I_{[-1,1]}(x).
\]
By symmetry, $f_2(y) = f_1(y)$ ($X$ and $Y$ have the same marginal distributions.)

(b) $X$ and $Y$ are not independent because $f(x, y) \neq f_1(x)f_2(y)$.

**Question 1**: Determine $P(Y = 1|X = 1)$ and $P(Y = 2|X = 1)$.

$P(Y = 1|X = 1) = 0.92 / 0.323 = 0.285$ and $P(Y = 2|X = 1) = 0.54 / 0.323 = 0.1671$.

**Question 2**: Determine $\Pr(X < 5|Y = 1)$ and $\Pr(X < 5|Y = 4)$.

$\Pr(X < 5|Y = 1) = 0.982$ and $\Pr(X < 5|Y = 4) = 0.632$. 