Page 65 Exercise 5: A box contains \( r \) red balls and \( b \) blue balls. One ball is selected at random and its color is observed. The ball is then returned to the box and \( k \) additional balls of the same color are also put into the box. A second ball is then selected at random, its color is observed, and it is returned to the box together with \( k \) additional balls of the same color. Each time another ball is selected, what is the probability that the first three balls will be red and the fourth ball will be blue?

Solution: Let \( R_i \) be the event that a red ball is drawn on the \( i \)th draw, and let \( B_i \) be the event that a blue ball is drawn on the \( i \)th draw for \( i = 1, \ldots, 4 \). Then

\[
Pr(R_1) = \frac{r}{r+b}, \\
Pr(R_2|R_1) = \frac{r+k}{r+b+k}, \\
Pr(R_3|R_2) = \frac{r+2k}{r+b+2k}, \\
Pr(B_4|R_2R_3) = \frac{b}{r+b+3k}.
\]

Then we want the product of these probabilities, namely

\[
Pr(R_1)(r+k)(r+2k)b \\
(\frac{r}{r+b})(\frac{r+b+k}{r+b+2k})(\frac{r+b+3k}{r+b+3k})
\].

Page 65 Exercise 12 (U): For any three events \( A, B, \) and \( D \), such that \( Pr(D) > 0 \), prove that

\[
Pr(A \cup B | D) = Pr(A | D) + Pr(B | D) - Pr(A \cap B | D).
\]

This is the conditional version of Theorem 1.5.7.

Proof. Let \( A_1 = A \cap D \) and \( A_2 = B \cap D \). Then \( A_1 \cup A_2 = (A \cup B) \cap D \) and \( A_1 \cap A_2 = A \cap B \cap D \). Now apply Theorem 1.5.7 to determine \( Pr(A_1 \cup A_2) \).

\[
Pr([A \cup B] \cap D) = Pr(A_2 \cup A_2) = Pr(A_1) + Pr(A_2) - Pr(A_1 \cap A_2)
\]

\[
= Pr(A \cap D) + Pr(B \cap D) - Pr(A \cap B \cap D).
\]

Now, we divide the extreme left and right ends of this string of equalities by \( Pr(D) \) to obtain:

\[
Pr([A \cup B] | D) = \frac{Pr([A \cup B] \cap D)}{Pr(D)}
\]

\[
= \frac{Pr(A \cap D) + Pr(B \cap D) - Pr(A \cap B \cap D)}{Pr(D)}
\]

\[
= Pr(A | D) + Pr(B | D) - Pr(A \cap B | D).
\]

Page 65 Exercise 13: A box contains three coins with a head on each side, four coins with a tail on each side, and two fair coins. If one of these nine coins is selected at random and tossed once, what is the probability that a head will be obtained?

Solution: Let \( A_1 \) denote the event that the selected coin has a head on each side, let \( A_2 \) denote the event that it has a tail on each side, let \( A_3 \) denote the event that it is fair, and let \( B \) denote the event that a head is obtained. Then

\[
Pr(A_1) = \frac{3}{9}, \quad Pr(A_2) = \frac{4}{9}, \quad Pr(A_3) = \frac{2}{9}, \\
Pr(B|A_1) = 1, \quad Pr(B|A_2) = 0, \quad Pr(B|A_1) = \frac{1}{2}.
\]
Hence,

\[ Pr(B) = \sum_{i=1}^{3} Pr(A_i) Pr(B|A_i) = \frac{4}{9} \]

Page 65 Exercise 17: (G) Assume \( Pr(B_j \cap C) > 0 \) for each \( j \), otherwise (2.1.5) is undefined. By applying the definition of conditional probability to each term, the right-hand side of (2.1.5) can be expressed as

\[
\sum_{j=1}^{k} \frac{Pr(B_j \cap C)}{Pr(C)} \cdot \frac{Pr(A \cap B_j \cap C)}{Pr(B_j \cap C)} = \frac{1}{Pr(C)} \sum_{j=1}^{k} \frac{Pr(A \cap B_j \cap C)}{Pr(C)} = \frac{Pr(A \cap C)}{Pr(C)}.
\]

Page 75 Exercise 10: The probability that any child in a certain family will have blue eyes is \( \frac{1}{4} \), and this feature is inherited independently by different children in the family. If there are five children in the family and it is known that at least one of these children has blue eyes, what is the probability that at least three of the children have blue eyes?

Solution: The probability \( p_j \) that exactly \( j \) children will have blue eyes is

\[ p_j = \binom{5}{j} \left( \frac{1}{4} \right)^j \left( \frac{3}{4} \right)^{5-j} \text{ for } j = 0, 1, \ldots, 5. \]

Hence the desired probability is:

\[ \frac{p_3 + p_4 + p_5}{p_1 + p_2 + p_3 + p_4 + p_5} \]

Page 76 Exercise 17: In order that the target is hit the first time by boy A on his third throw, A must miss on his first 2 throws and B miss on his first two throws. Throws are independent so the probability of all 5 of these events happening is the product

\[ \frac{2}{3} \times \frac{3}{4} \times \frac{2}{3} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{12}. \]

Page 76 Exercise 18: Two boys A and B throw a ball at a target. Suppose that the probability that boy a will hit the target on any throw is \( \frac{1}{3} \) and the probability that boy B will hit the target on any throw is \( \frac{1}{4} \). Suppose also that boy A throws first and the two boys take turns throwing. Determine the probability that boy A will hit the target before boy B does.

Solution: Let \( E \) denote the event that boy A hits the target before boy B. There are two methods of solving this problem. The first method is to note that the event \( E \) can occur in two different ways: (i) If A hits the target on the first throw. This event occurs with probability \( \frac{1}{3} \). (ii) If both A and B miss the target on their first throws, and then subsequently A hits the target before B. The probability that A and B will both miss on their first throws is \( \frac{2}{3} \times \frac{3}{4} = \frac{1}{2} \). Then they do miss, the conditions of the game become exactly the same as they were at the beginning of the game. In effect, it is as if the boys were starting a new game all over again, and so the probability that A will subsequently hit the target before B is again \( Pr(E) \). Therefore, by considering these two ways in which the event \( E \) can occur, we obtain the relation

\[ Pr(E) = \frac{1}{3} + \frac{1}{2} Pr(E). \]

The solution is \( Pr(E) = \frac{2}{3} \).

The second method extends the calculation in Problem 17 to obtain

\[ Pr(E) = \frac{1}{3} \sum_{j=1}^{\infty} \left( \frac{1}{2} \right)^{j-1} = \frac{1}{3} \cdot 2 = \frac{2}{3}. \]
Page 76 Exercise 23: A programmer is about to attempt to compile a series of 11 similar programs. Let $A_i$ be the event that the $i$th program compiles successfully for $i = 1, \ldots, 11$. When the programming task is easy, the programmer expects that 80% of programs should compile. When the programming task is difficult, she expects that only 40% of the programs will compile. Let $B$ be the event that the programming task was easy. The programmer believes that the events $A_1, \ldots, A_{11}$ are conditionally independent given $B$ and given $B^c$.

(a): Compute the probability that exactly 8 out of 11 programs will compile given $B$.

Solution: Conditional on $B$, the events $A_1, \ldots, A_{11}$ are independent with probability 0.8 each. The conditional probability that a particular collection of eight programs out of the 11 will compile is $(0.8)^8(0.2)^3 = 0.001342$. There are $\binom{11}{8} = 165$ different such collections of eight programs out of the 11, so the probability that exactly eight programs will compile is $165 \cdot 0.001342 = 0.2215$.

(b): Compute the probability that exactly 8 out of 11 programs will compile given $B^c$.

Solution: Conditional on $B^c$ the events $A_1, \ldots, A_{11}$ are independent with probability 0.4 each. The conditional probability that a particular collection of eight programs out of the 11 will compile is $(0.4)^8(0.6)^3 = 0.0004116$. There are $\binom{11}{8} = 165$ different such collections of eight programs out of the 11, so the probability of exactly eight programs compiling is $0.02335$.

Page 85 Exercise 6: Suppose that when a machine is adjusted properly, 50 percent of the items produced by it are of high quality and the other 50 percent are of medium quality. Suppose, however, that the machine is improperly adjusted during 10 percent of the time and that, under these conditions, 25 percent of the items produced by it are of high quality and 75 percent are of medium quality.

(a): Suppose that five items produced by the machine at a certain time are selected at random and inspected. If four of these items are of high quality and one item is of medium quality, what is the probability that the machine was adjusted properly at that time?

Solution: Let $A_1$ denote the event that the machine is adjusted properly, let $A_2$ denote the event that it is adjusted improperly, and let $B$ be the event that four of the five inspected items are of high quality. Then:

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

$$= \frac{0.9(\binom{5}{4})0.5^5}{0.9(\binom{5}{4})0.5^5 + 0.1(\binom{5}{4})0.25^40.75} = \frac{96}{97}$$

(b): Suppose that one additional item, which was produced by the machine at the same time as the other five items, is selected and found to be of medium quality. What is the new posterior probability that the machine was adjusted properly?

Solution: The prior probabilities before this additional item is observed are the values found in part (a): $P(A_1) = \frac{96}{97}$ and $P(A_2) = \frac{1}{97}$. Let $C$ denote the event that the additional item is of medium quality. Then:

$$P(A_1|C) = \frac{\frac{96}{97} \frac{1}{2} - \frac{1}{3}}{\frac{96}{97} \frac{1}{2} + \frac{1}{97} \frac{1}{4}} = \frac{64}{65}$$

Page 85 Exercise 8: Consider again the box containing the five different coins described in Exercise 7. Suppose that one coin is selected at random from the box and is tossed repeatedly until a head is obtained.

(a): If the first head is obtained on the fourth toss, what is the posterior probability that the $i$th coin was selected ($i = 1, \ldots, 5$)?
Solution: If coin \( i \) is selected, the probability that the first head will be obtained on the fourth toss is \((1 - p_i)^3 p_i\). Therefore, the posterior probability that coin \( i \) was selected is

\[
\pi_i = \frac{\frac{1}{5} (1 - p_i)^3 p_i}{\sum_{j=1}^{5} \pi_j (1 - p_j)^3 p_j} \quad \text{for } i = 1, \ldots, 5
\]

(b): If we continue to toss the same coin until another head is obtained, what is the probability that exactly three additional tosses will be required?

Solution: If coin \( i \) is used, the probability that exactly three additional tosses will be required to obtain another head is \((1 - p_i)^2 p_i\). Therefore, the desired probability is

\[
\sum_{i=1}^{5} \pi_i (1 - p_i)^2 p_i = 0.1291.
\]

Page 85 Exercise 10: Returning to example 2.3.5, for which values of \( i \) is the posterior probability that the parents have the genotypes of the event \( B_i \) smaller than the prior probabilities that the parents have the genotypes of the event \( B_i \)?

The posterior probabilities are computed according to \( \Pr(B_i|E) = \Pr(E|B_i) \Pr(B_i)/\Pr(E) = \Pr(E|B_i) \Pr(B_i) \times \frac{3}{4} \), since \( \Pr(E) = 3/4 \). Thus,

\[
\begin{align*}
\Pr(B_1|E) &= \frac{1}{16} \times 1 \times \frac{3}{4} = \frac{1}{12} \\
\Pr(B_2|E) &= \frac{1}{4} \times 1 \times \frac{4}{3} = \frac{1}{3} \\
\Pr(B_3|E) &= \frac{1}{8} \times 4 \times \frac{1}{3} = \frac{1}{6} \\
\Pr(B_4|E) &= \frac{1}{4} \times \frac{3}{4} \times \frac{4}{3} = \frac{1}{4} \\
\Pr(B_5|E) &= \frac{1}{4} \times \frac{1}{2} \times \frac{4}{3} = \frac{1}{6} \\
\Pr(B_6|E) &= 0
\end{align*}
\]

The prior probabilities are listed in the example, and by inspection the posterior probabilities for \( B_5 \) and \( B_6 \) are smaller.

Solutions for In-notes questions of Chapter 2

Question 1 : Determine \( \Pr(B|A) \Pr(B|A) = \frac{8}{50} = .16 < .23 = \Pr(B) \).

Question 2 : What is the probability of obtaining 2 red balls in the experiment above (that is, supposing that 3 balls are drawn randomly and without replacement)?

The outcomes of the sampling experiment and their associated probabilities are tabulated:
From the table, the probability function of \( X \) is determined to be

\[
\begin{array}{c|c}
 x & f(x) \\
0 & \frac{336}{720} \\
1 & \frac{48}{720} \\
2 & \frac{16}{720} \\
\hline
0 & \text{for } x \neq 0, 1, 2
\end{array}
\]

Alternatively, since sampling is carried out randomly and without replacement from a population of 10 units, 2 two of which are red and 8 are green,

\[
f(x) = \begin{cases} 
\frac{\binom{2}{x} \binom{8}{3-x}}{\binom{10}{3}}, & \text{if } x \in \{0, 1, 2\}, \\
0, & \text{otherwise.}
\end{cases}
\]

**Question 3**: Prove \( \Pr(A \cup B|C) = \Pr(A|C) + \Pr(B|C) - \Pr(AB|C) \).

\[
\Pr(A \cup B|C) = \frac{\Pr((A \cup B) \cap C)}{\Pr(C)}
\]

\[
= \frac{\Pr(AC \cup BC)}{\Pr(C)}
\]

\[
= \frac{\Pr(AB^cC) + \Pr(A^cBC) + \Pr(ABC)}{\Pr(C)}
\]

\[
= \frac{\Pr(AC) + \Pr(A^cBC)}{\Pr(C)}
\]

\[
= \frac{\Pr(AC) + \Pr(B) - \Pr(ABC)}{\Pr(C)}
\]

\[
= \Pr(A|C) + \Pr(B|C) - \Pr(A \cap B|C).
\]