Confidence intervals
D. Patterson, Dept. of Mathematical Sciences, U. of Montana

There are four main ways we will learn to construct a confidence interval for an unknown parameter $\theta$ based on a random sample $X_1, \ldots, X_n$ from $f(x|\theta)$.

1. Exact, using a pivotal quantity. A function $g(X, \theta)$ is said to be a pivotal quantity if its distribution does not depend on the parameter $\theta$. To construct a $\gamma\%$ confidence interval for $\theta$, find numbers $c_1$ and $c_2$ such that $P(c_1 < g(X, \theta) < c_2) = \gamma$. Usually we take $c_1$ and $c_2$ so that there is equal area in both tails; that is, $c_1$ is the $\alpha/2$ quantile of the distribution of $g(X, \theta)$ and $c_2$ is the $1 - \alpha/2$ quantile where $\alpha = 1 - \gamma$. Then solve for $\theta$ to get the confidence interval. We’ve covered several situations where this is possible (for ease of notation below, let $\alpha = 1 - \gamma$).

(a) Mean of a normal distribution, standard deviation known. A pivotal quantity is

$$g(X, \mu) = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

A 100$\gamma\%$ confidence interval for $\mu$ is

$$\bar{X} \pm c \frac{\sigma}{\sqrt{n}}$$

where $c$ is the $1 - \alpha/2$ quantile of the $N(0,1)$ (because of the symmetry of the $N(0,1)$, we have $c_1 = -c_2$).

(b) Mean of a normal distribution, standard deviation unknown. A pivotal quantity is

$$g(X, \mu) = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

where $S = \sum(X_i - \bar{X})^2/(n - 1)$ is the sample standard deviation (denoted $\sigma'$ in the text). A 100$\gamma\%$ confidence interval for $\mu$ is

$$\bar{X} \pm c \frac{S}{\sqrt{n}}$$

where $c$ is the the $1 - \alpha/2$ quantile of the $t_{n-1}$ distribution.

(c) Variance of a normal distribution, mean known. A pivotal quantity is

$$\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\sigma^2} \sim \chi^2_n.$$

A 100$\gamma\%$ confidence interval for $\sigma^2$ is

$$\left(\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{c_2}, \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{c_1}\right)$$

where $c_1$ is the $\alpha/2$ quantile and $c_2$ is the $1 - \alpha/2$ quantile of $\chi^2_n$ distribution.
(d) Variance of a normal distribution, mean unknown. A pivotal quantity is
\[ \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}. \]
A 100\(\gamma\)% confidence interval for \(\sigma^2\) is
\[ \left( \frac{\sum_{i=1}^{n} (X_i - \mu)^2}{c_2}, \frac{\sum_{i=1}^{n} (X_i - \mu)^2}{c_1} \right) \]
where \(c_1\) is the \(\alpha/2\) quantile and \(c_2\) is the \(1 - \alpha/2\) quantile of \(\chi^2_{n-1}\) distribution.

(e) Mean of an exponential distribution (problem 6, p. 415). Recognizing that an exponential random variable with mean \(\mu\) is Gamma(1,1/\(\mu\)), by Theorem 5.9.3 on p. 298 and problem 1 on p. 301, a pivotal quantity is
\[ \frac{1}{\mu} \sum_{i=1}^{n} X_i \sim \text{Gamma}(n,1). \]
A 100\(\gamma\)% confidence interval for \(\mu\) is
\[ \left( \frac{\sum_{i=1}^{n} X_i}{c_2}, \frac{\sum_{i=1}^{n} X_i}{c_1} \right) \]
where \(c_1\) is the \(\alpha/2\) quantile and \(c_2\) is the \(1 - \alpha/2\) quantile of a Gamma(n,1) distribution.

2. If the parameter of interest is the mean \(\mu\), then we can use the Central Limit Theorem to derive an approximate confidence interval for \(\mu\). First, by the CLT, \(\sqrt{n}(\bar{X}_n - \mu)/\sigma \overset{d}{\to} \text{N}(0,1)\). Then, since the sample standard deviation \(S\) is a consistent estimator of \(\sigma\) for any distribution (that is, \(S \overset{p}{\to} \sigma\)), we have (by Slusky’s Theorem) that \(\sqrt{n}(\bar{X}_n - \mu)/S \overset{d}{\to} \text{N}(0,1)\). Using this as the pivotal quantity, it follows that an approximate confidence interval for \(\mu\) is
\[ \bar{X}_n \pm c \frac{S}{\sqrt{n}} \]
where \(c\) is from a N(0,1) distribution. Sometimes, we don’t need to “plug-in” \(S\) for \(\sigma\). If \(\sigma\) is a function of \(\mu\) (as with the binomial and Poisson distributions), then you can derive an approximate confidence interval for \(\mu\) without making the substitution.

3. More generally, if certain regularity conditions are satisfied, we can use the results on pp. 442-3 to derive an approximate confidence interval for any parameter \(\theta\) based on its MLE \(\hat{\theta}\). By the results there,
\[ \frac{\hat{\theta} - \theta}{\sqrt{1/nI(\theta)}} \overset{d}{\to} \text{N}(0,1). \]
Hence, an approximate confidence interval for \(\theta\) is
\[ \hat{\theta} \pm c \left( \frac{1}{nI(\theta)} \right)^{1/2} \]
where $c$ is from a $N(0,1)$ distribution. However, $I(\theta)$ usually depends on $\theta$. Therefore, we can do as in item 3 and substitute $\hat{\theta}$ for $\theta$ in $I(\theta)$ since the MLE is consistent. Hence, the confidence interval becomes

$$\hat{\theta} \pm c \sqrt{\frac{1}{nI(\hat{\theta})}}.$$  

4. Bootstrapping. This is discussed in Section 11.5.

Notes

- The confidence interval for a parameter $\theta$ often takes the form

$$\hat{\theta} \pm c \text{SE}(\hat{\theta})$$

where $\text{SE}(\hat{\theta})$ is an estimate of the standard deviation of the sampling distribution of $\hat{\theta}$ and is called the **standard error** of $\hat{\theta}$.

- Confidence intervals for a parameter $\theta$ can be transformed to a confidence interval for any continuous invertible function of $\theta$, say $g(\theta)$. If $L$ and $U$ are the lower and upper limits for a $100\gamma\%$ confidence interval for $\theta$, then $g^{-1}(L)$ and $g^{-1}(U)$ are the limits of a $100\gamma\%$ confidence interval for $g(\theta)$. For example, in 1(e), the exact confidence interval for $\mu$ in an exponential distribution can be transformed to an exact confidence interval for the parameter $\beta = 1/\theta$:

$$\left(\frac{c_1}{\sum_{i=1}^{n} X_i}, \frac{c_2}{\sum_{i=1}^{n} X_i}\right).$$

For exact confidence intervals, it does not matter whether we get an exact confidence interval for $\theta$ and transform to an exact confidence interval for $g(\theta)$ or whether we find an exact confidence interval for $g(\theta)$ and transform to an exact confidence interval for $\theta$, we’ll get the same result (e.g., in 1(e), we could get an exact confidence interval for $\beta$ and transform to an exact confidence interval for $\mu$). However, if the confidence interval for $\theta$ is based on an approximate result as in 2 and 3 above, then it can make a difference: for example, do we get an approximate confidence interval for $\mu$ and transform to get a confidence interval for $\theta = 1/\mu$ or do we get an approximate confidence interval for $\theta$ and transform to get an approximate confidence interval for $\mu = 1/\theta$? It’s best to start with the one whose estimator is more closely normal. That may not be obvious. Bootstrapping is a way to address this in two ways: one, it allows you to estimate the shape of the sampling distribution of an estimator, and, two, it can create confidence intervals nonparametrically which avoids this problem.

- Exact confidence intervals, if they exist, are exact for any sample size $n$ while asymptotic confidence intervals may perform poorly for small $n$. What does it mean for a confidence interval procedure to “perform poorly”? It means that you assume that $95\%$ of $95\%$ confidence intervals contain the true value of the parameter, but it may be that only $90\%$ or $80\%$ or
fewer of approximate confidence intervals actually do (or it may be more than 95% which isn’t so bad, but means that your confidence intervals are wider than they need to be). Thus you act with more confidence than is warranted. It’s usually impossible to tell how well approximate confidence intervals perform in reality without simulation. The actual coverage of the intervals usually depends on the value of the parameter and the sample size.

- Exact confidence intervals are valid only if the assumed model is true. If the model is wrong, then the actual coverage of the confidence intervals may not be what is reported. The robustness of the confidence interval procedure to the model assumptions is usually investigated by simulation. The same caveat applies to asymptotic confidence intervals for parameters of any model. One partial exception are confidence intervals for a mean where the standard deviation is estimated by the sample standard deviation, since the CLT is valid for any distribution and the sample standard deviation is a consistent estimate of the population standard deviation for any distribution. You still have to worry about whether the sample size is big enough for the approximate normality to hold.

- The t confidence interval for the mean of a normal population has been found to be very robust to the assumption of normality. That is, the resulting confidence intervals have very nearly the desired coverage even for very non-normal distributions and for small sample sizes. Generally, unless the population distribution is very skewed, the t confidence intervals achieve the claimed coverage for samples sizes as small as $n = 10$ or 15; even for skewed distributions, a sample size of $n = 30$ often achieves the desired coverage.