1. For each positive integer \( n \), let \( R(n) \) be the integer closest to \( \sqrt{n} \). For example, 
\[
R(1) = 1 = R(2), \quad R(3) = 2 = R(4) = R(5) = R(6), \quad R(7) = 3 = R(8) = \cdots = R(12).
\] 
- Show the definition of \( R(n) \) is never ambiguous. That is, show there is no integer \( n \) whose square-root is not an integer but which is midway between two consecutive integers.
- Compute the exact value of \( \sum_{j=1}^{2004} \frac{1}{R(j)} \).

2. Alice and Ben drive for 100 miles in separate cars. The speed limit for the first 50 miles is 55 miles-per-hour, and 65 miles-per-hour for the second 50 miles. Alice stops for gas during the first (50 mile) leg of the trip and Ben stops for gas during the second leg; both stops take 15 minutes. Assume each person drives at the posted speed limits, other than while stopped, for both parts of their whole trip.
- Which person finishes first? Explain.
- Analyze a slightly different scenario. Suppose first leg of the trip is 40 miles and the second leg is 60 miles. Which person finishes first? Explain.
- Suppose both legs are 50 miles, but the first leg has an even lower speed limit while the second leg has an even larger speed limit. Which person finishes first? Explain.

3. Find the remainder when the polynomial 
\[
p(x) = 1 + x + x^5 + x^{125} + x^{625}
\]
is divided by \( x^2 - 1 \)?

4. The following figures share several properties. A, B, C, D are vertices of a square whose sides have length 1. W, X, Y, Z are points on sides of that square such that line segments AW, BX, CY, DZ have length \( \lambda \). Lines AX, BY, CZ, DW meet at points a, b, c, d.

Compute area of the quadrilateral whose vertices are a, b, c, d. 
Note: that area is a function of \( \lambda \).

5. Suppose \( A \) is an \( n \times n \) matrix each of whose entries is either 1 or \(-1\). Show the determinant of \( A \) is divisible by \( 2^{n-1} \).
6. Suppose \( g \) is a smooth function whose first and second derivatives are continuous on the interval \([0, 1]\); also suppose values of \( g \) are positive and the graph of \( g \) is concave down on that domain interval. Let \( L(a) \) be the line tangent to the graph of \( g \) at the point \((a, g(a))\) and let \( T(a) \) be the area of the trapezoid bounded by the \( y\)-axis on the left, the vertical line \( x = 1 \) on the right, the \( x\)-axis on the bottom, and line \( L(a) \) on the top.

For each such function \( g \), there is a choice of \( a \) in \([0, 1]\) where the trapezoid area function \( T(a) \) attains its minimum value. Locate that value of \( a \) and explain why your answer is correct.

7. The highway between cities A and B is 999 miles long. Along the way from A to B, at one mile intervals, are signs indicating the distances (in miles) to A and B, respectively:

\[
(0 : 999), \quad (1 : 998), \quad (2 : 997), \quad (3 : 996), \quad \ldots, \quad (998 : 1), \quad (999 : 0).
\]

The first sign involves two digits (‘0’ and ‘9’); the second sign involves three digits (‘1’, ‘8’, and ‘9’). There are 1000 signs — how many involve exactly two different digits?

8. Are there any non-trivial polynomial functions which are equal to the product of their first and second derivatives? In other words, is there a non-trivial polynomial \( p(x) \) such that

\[
p(x) = p'(x) \cdot p''(x)
\]

for all \( x \)? If NO, then explain why not. If YES, then find one or several and discuss whether you have found all of them.

9. Consider two functions whose domain is the set of positive integers. Let \( A(n) \) count the ways to write \( n \) as a sum whose terms are either 1 or 2, ignoring the ordering of the terms. For example, \( A(4) = 3 \) because

\[
4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 2 + 2.
\]

Let \( B(n) \) be the number of ways when the ordering matters. For example, \( B(4) = 5 \) because

\[
4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 2 + 1 = 2 + 1 + 1 = 2 + 1 + 1.
\]

The following table shows a few values of these functions.

<table>
<thead>
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<th>( n )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
</tr>
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<tbody>
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<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( B(n) )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

- Compute \( A(6) \) and \( B(6) \).
- Find a general way to compute \( A(n) \). Show your method is correct.
- Find a general way to compute \( B(n) \). Show your method is correct.

10. Consider the sequence whose \( n^{th} \) term is \( \theta_n = \arctan(n) \). Prove that

\[
\theta_{n+1} - \theta_n < \frac{1}{n + n^2}
\]

for each positive integer \( n \).

**Note:** \( \arctan = \tan^{-1} \), the inverse-tangent function, has domain \((-\infty, \infty)\) and range \((-\frac{\pi}{2}, \frac{\pi}{2})\).