1. Consider the set of fractions $\frac{n}{2002}$ where $n$ is an integer such that $1 \leq n \leq 2001$. For some choices of $n$, the fraction $\frac{n}{2002}$ can be reduced, i.e., rewritten as the ratio of smaller integers. For instance

$$\frac{7}{2002} = \frac{1}{286}, \quad \frac{11}{2002} = \frac{1}{182}, \quad \frac{13}{2002} = \frac{1}{154}, \quad \frac{78}{2002} = \frac{6}{154}.$$ 

How many of the fractions $\frac{n}{2002}$ can not be reduced?

2. The Great Eulerdini claims profound powers have come from her intense studies of secret mathematical documents. She recently asked me to shuffle an ordinary deck of cards and place them face down on a table. She held my hands and chanted some words that I did not understand but which she said invoked magical powers — then The Great Eulerdini had me move the cards into two separate but equal-size sets (without either of us seeing the face-side of any card). She proclaimed that her magic had forced the number of red cards in one set to be the same as the number of black cards in the other set. We turned the cards over and counted — lo and behold, the numbers matched — just as she said they would!

Explain why this card trick works.

3. $A$, $B$, $C$ are equilateral triangles such that

$$\text{area}(A) + \text{area}(B) = \text{area}(C).$$

How are the side-lengths of these triangles related? Explain.

4. Recall that “integration by parts” is just the “product rule in reverse”. A traditional statement of integration-by-parts has the form

$$\int u \, dV = u \cdot V - \int V \, du.$$ 

**Explain why the following work goes wrong.**

If $u = x^{-1}$ and $dV = dx$, then $du = -x^{-2} \, dx$ and $V = x$. Therefore

$$\int \frac{1}{x} \, dx = \frac{1}{x} \cdot x - \int x \cdot \frac{dx}{x^2} = 1 - \int \frac{x}{x^2} \, dx = 1 + \int \frac{1}{x} \, dx.$$ 

Subtract $\int \frac{1}{x} \, dx$ from both sides to infer

$$0 = 1.$$ 

5. $47 \times 74 = 74 \times 47$ and $26 \times 93 = 62 \times 39$.

The first equation seems trivial, but the second may be a surprise.

How many pairs of two-digit numbers have the property that the value of their product is not changed by interchanging the digits of each number in the pair?

6. How can a square be doubled or halved without doing measurements? For each part of this problem, imagine you are writing a letter to a friend with instructions that specify a sequence of operations involving folding, creasing, or cutting some paper or joining some pieces of paper with tape.

- Start with two equal-size squares. Produce a square whose area is double the area of one of the original squares.
- Start with one square. Produce a square whose area is half of the original square.
7. Underlined letters in the left-hand tableau spell the name of a famous American who came to Montana; the path of bold letters spells the same name and so does the path of uppercase letters.
- How many paths in the left-hand tableau spell the name of that renowned explorer?
- The right-hand tableau combines the family names of Meriwether Lewis and William Clark. How many paths in that tableau spell “LewisClark”?
- Write a short proof that both of your counts are correct.

\[
\begin{array}{c c c c c c c c c c c}
L & E & w & i & s & C & l & a & r & k \\
 e & W & i & S \\
 w & I & S \\
i & s \\
s & \\
\end{array}
\]

\[
\begin{array}{c c c c c c c c c c c}
L & e & w & i & s & C & l & a & r & k \\
e & w & i & s & C & l & a & r & k \\
w & i & s & C & l & a & r & k \\
i & s & C & l & a & r & k \\
s & C & l & a & r & k \\
C & l & a & r & k \\
l & a & r & k \\
ar & k \\
r & k \\
k & \\
\end{array}
\]

8. Suppose \(a\) and \(b\) are different positive integers. Is it possible for
\[
\frac{a}{b} + \frac{b}{a}
\]
to be an integer? Prove your answer is correct.

9. Four circles are arranged so each is tangent to two others. (Two such arrangements are shown below.) Prove the four points of tangency also lie on a circle (shown here as a thin dotted curve).

10. Suppose \(f\) and \(g\) are differentiable functions with the same domain. In Calculus I, you learn some symbolic rules for differentiating a sum or product of such functions.
\[
(f + g)' = f' + g'
\]
\[
(f \cdot g)' = (f' \cdot g) + (f \cdot g')
\]
Some students struggle to “unlearn” the

\textbf{Naive Product Rule :} \quad (f \cdot g)' = f' \cdot g'

because it seems temptingly simple even though it is not always true. Notwithstanding the fact that the Naive Product Rule is not an identity, this problem asks you to find non-constant differentiable functions \(f\) and \(g\) for which the Naive Product Rule happens to yield the correct result.