INSTRUCTIONS: There is no required minimum number of problems to be solved but please do as many of the following problems as time permits. Quality work is more important than quantity and it is not expected that all problems be attempted; five solutions might be a reasonable goal. Succinct solutions displaying insight are most desirable. Partial solutions and/or commentary are also encouraged. Calculators are permitted.

1. Cucumbers are assumed, for present purposes, to be a substance that is 99% water by weight. If 500 pounds of cucumbers are allowed to stand overnight, and if the partially evaporated substance that remains in the morning is 98% water, how much is the morning weight?

2. If m and n are odd integers, prove that the equation
   \[ x^2 + 2mx + 2n = 0 \]
   has no rational roots.

3. How many zeroes does 1,000,000! (one million factorial) end in?

4. Prove that for all natural numbers n:
   \[ \sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}. \]

5. The year 1994 is the 120\textsuperscript{th} anniversary of the birthdate of N. J. Lennes (1874-1951).
   a) Determine the largest prime factor p in 1874 and the largest prime factor q in 1994.
   b) Determine all the prime factors between this p and q.
   c) For a real number x, the function that returns the next prime number greater than or equal to x is denoted (e.g., in the software DERIVE\textsuperscript{TM}) by NEXT\_PRIME(x). Sketch a graph of the NEXT\_PRIME function for \(-10 \leq x \leq +10\).
   d) Sketch and discuss what the graph of NEXT\_PRIME would look like on your computer screen for \(-10,000 \leq x \leq +10,000\). Take into account that there are arbitrarily long gaps in which there are no prime numbers; e.g., if B stands for a billion then B! + 2, B! + 3, \ldots, B! + B is a very long gap with no primes.

6. Show that there exists a positive number \(\lambda\) such that
   \[ \int_{0}^{\pi} x^\lambda \cdot \sin(x) \, dx = 3. \]
7. A railroad track runs absolutely straight and level for exactly one mile (the curvature of the earth having been flattened). Assume that its two ends remain fixed, but one additional foot of track is inserted in the middle and then seamlessly welded in with the rest. Assume, moreover, that the track buckles up in the shape of an arc of a circle. Question: how far is the middle of that arc off the ground?

8. Prove that \( \cos \left( \frac{\pi x}{2} \right) \leq 1 - x^2 \) for every \( x \) in the closed interval \([0, 1]\).

9. A strip of paper tied into a knot and pulled tightly and trimmed forms a regular pentagon. If the strip of paper is 1 inch wide, what is the area of the pentagon?

10. A drunkard begins at home and walks toward his favorite bar. When he gets halfway there he changes his mind and turns around to return home. After he walks half of the way toward home, he again changes his mind and turns around toward the bar. He walks halfway the distance to the bar and turns around again. He continues this indecision, each time walking halfway toward his destination. Describe the man’s fate if he continues walking in this manner into the infinity of time. (A variation of this problem appeared in the “Ask Marilyn” column of Parade.)

11. Let \( f \) be a continuous function on the reals, and define \( F \) on the reals by
\[
F(x) = \int_{1/x}^{x} f(t) \, dt.
\]
Find the derivative \( F' \) of \( F \).

12. Ten (not necessarily all different) integers have the property that if all but one of them are added, the possible results (depending on which one is omitted) are: 82, 83, 84, 85, 87, 89, 90, 91, 92. (That is not a misprint; there are only nine possible results.) What are the ten integers?

13. Although the orbits of Venus and Earth are elliptical with the Sun at a focus, the eccentricity is close enough to 1 that you can assume the orbit is a circle with the Sun at the center. Suppose that Venus is between the Earth and the Sun. If the orbit of Venus is 224 days and the orbit of the Earth is 365.25 days, how many days until Venus is next between the Earth and the Sun?

14. If \( a, b \) and \( c \) are positive numbers whose sum is 1, then prove and generalize:
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.
\]