Chapter 11

The Structure of the Real Line

We now understand the algebraic properties of $\mathbb{R}$. In particular, we know that $\mathbb{R}$ satisfies the ordered field axioms. We also know that $\mathbb{R}$ is complete, and that completeness is a property of a space that allows us to develop a satisfying theory of limits of sequences because it gives us a necessary and sufficient condition for convergence.

We now want to explore additional properties of $\mathbb{R}$ that will be important as we develop a theory of functions from $\mathbb{R}$ to $\mathbb{R}$. In particular, we want to talk about limits of functions and properties of continuous functions. Consider, for example, the following theorem, which should be familiar from calculus.

**Theorem 11.0.1.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then $f$ attains a maximum and minimum value, i.e., there exists $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$ and there exists $d \in [a, b]$ such that $f(d) \leq f(x)$ for all $x \in [a, b]$.

This theorem is a key component of the procedure we use in calculus to find the maximum of a function on a closed and bounded interval $[a, b]$.

In Theorem 11.0.1, is the form of the set on which $f$ is continuous important? Would the same theorem hold if, for example, the hypothesis were that $f$ is continuous on the open interval $(a, b)$? The answer is No. Consider, for example, the function $f(x) = \frac{1}{x}$. We will show in the next chapter that $f$ is continuous on (for example) $(0, 1)$. This function, however, attains neither a maximum nor a minimum value of this set; it is unbounded, so it does not attain a maximum value. Also, although it is bounded below, the infimum of its image set is 1, but there is no $d \in (0, 1)$ for which $f(d) = 1$, and so the function does not attain a minimum value.

What property or properties does the set $[a, b]$ have that the set $(0, 1)$ does not have? In this chapter we explore this and other related questions about the structure of subsets of the real line.
11.1 Topological Structure

11.1.1 Open and closed sets

In earlier courses, you have met open and closed intervals. Recall, if \( a < b \), \( (a,b) = \{ \, x \in \mathbb{R} : a < x < b \, \} \) and \( [a,b] = \{ \, x \in \mathbb{R} : a \leq x \leq b \, \} \). The first set, the open interval, has the property that every point in the interval is completely surrounded by points of the interval. The second set does not have this property; there are two points \((a, b)\) for which every little interval around them contains both points in the interval and points not in the interval.

The next definition generalizes the notion of an open interval and also makes precise what it means for every point in a set to be completely surrounded by other points of the set.

**Definition 11.1.1.** Let \( U \subseteq \mathbb{R} \). \( U \) is **open** if, for every \( x \in U \) there exists \( \varepsilon > 0 \) such that \( (x-\varepsilon, x+\varepsilon) \subseteq U \).

It will not surprise you to hear that open intervals are examples of open sets.

**Proposition 11.1.1.** Suppose \( a < b \). Then \((a, b)\) is open.

**Proof.** Let \( x \in (a,b) \) so that \( a < x < b \). Let \( \varepsilon = \min\{b-x, x-a\} \). We claim that \((x-\varepsilon, x+\varepsilon) \subseteq (a,b)\). Indeed, let \( y \in (x-\varepsilon, x+\varepsilon) \). We must show \( y \in (a,b) \).

Observe,
\[
\begin{align*}
y &> x - \varepsilon \\
&= x - \min\{b-x, x-a\} \\
&\geq x - (x-a) = a.
\end{align*}
\]
Similarly,
\[
\begin{align*}
y &< x + \varepsilon \\
&= x + \min\{b-x, x-a\} \\
&\leq x + (b-x) = b.
\end{align*}
\]

**Exercise 11.1.1.** Show that both \( \emptyset \) and \( \mathbb{R} \) are open subsets of \( \mathbb{R} \).

**Exercise 11.1.2.** Show that \([0,1) = \{ \, x \in \mathbb{R} : 0 \leq x < 1 \, \}\) is not open.

**Exercise 11.1.3.** Show that \((-\infty, a) = \{ \, x \in \mathbb{R} : x < a \, \} \) and \((a, \infty) = \{ \, x \in \mathbb{R} : x > a \, \} \) are both open sets.

**Exercise 11.1.4.** Let \( x \in \mathbb{R} \). Show that \( \{x\} \) is not open.

It is important to know what set operations preserve the class of open sets.

**Proposition 11.1.2.** (a) If \( U_{\alpha}, \alpha \in \mathcal{A} \) are open, \( \bigcup_{\alpha \in \mathcal{A}} U_{\alpha} \) is open.
(b) If $U_j$, $1 \leq j \leq n$ are open, \( \cap_{j=1}^{n} U_j \) is open.

Proof. For part (a), let $x \in \cup_{\alpha \in A} U_{\alpha}$. Then there exists $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. Because $U_{\alpha_0}$ is open, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U_{\alpha_0}$.

Because $U_{\alpha_0} \subseteq \cup_{\alpha \in A} U_{\alpha}$, $(x - \varepsilon, x + \varepsilon) \subseteq \cup_{\alpha \in A} U_{\alpha}$.

For part (b), let $x \in \cap_{j=1}^{n} U_j$, i.e., $x \in U_j$ for all $1 \leq j \leq n$. For each $j$, because $U_j$ is open, there exists $\varepsilon_j > 0$ such that $(x - \varepsilon_j, x + \varepsilon_j) \subseteq U_j$.

Set $\varepsilon = \min\{\varepsilon_j : 1 \leq j \leq n\}$ and note that $\varepsilon$ is positive. Because $\varepsilon \leq \varepsilon_j$, $(x - \varepsilon, x + \varepsilon) \subseteq U_j$ for all $j$ and so $(x - \varepsilon, x + \varepsilon) \subseteq \cap_{j=1}^{n} U_j$.

Our proof of (b) can not be generalized to show that an intersection of countably many open sets is open; if $x \in \cap_{j=1}^{\infty} U_j$ and each $u_j$ is open, it is still the case that there exists $\varepsilon_j > 0$ such that $(x - \varepsilon_j, x + \varepsilon_j) \subseteq U_j$. However, if we continue to the next step of the proof, we encounter a problem. The set \{\varepsilon_j : j \in \mathbb{N}\} may not have a minimum element, and its infimum may be zero. In this case, we would not have a positive $\varepsilon$ for which $(x - \varepsilon, x + \varepsilon)$ is contained in the intersection. Our failure to generalize this proof method to the case of infinitely many sets does not imply that the result fails for infinitely many sets. But in this case, the result in fact is not true in general for infinitely many sets. The next example illustrates.

Example 11.1.1. Let $U_j = ( -1/j, 1/j )$. Each $U_j$ is open because it is an open interval. But $\cap_{j=1}^{\infty} U_j = \{0\}$, which is not open by Exercise 11.1.4.

We turn our attention now to closed sets.

Definition 11.1.2. Let $E \subseteq \mathbb{R}$. $E$ is closed if $E^c$ is open.

As a first elementary example, we observe that the closed interval $[a, b]$ is, in fact, a closed set. Indeed,

$[a, b]^c = ( -\infty, a ) \cup ( b, \infty )$.

By Exercise 11.1.3, each set on the right is open, and by Proposition 11.1.2, their union is open. Because $[a, b]^c$ is open, by definition, $[a, b]$ is closed.

Exercise 11.1.5. Show that both $\emptyset$ and $\mathbb{R}$ are closed.

Exercise 11.1.6. Show that any finite set $\{x_1, \ldots, x_n\}$ is closed.

Exercise 11.1.7. Show that $[0, 1)$ is not closed.

Our next proposition is the analogue of Proposition 11.1.2 for closed sets. We leave the proof to the reader.

Proposition 11.1.3. (a) If $E_{\alpha}$, $\alpha \in A$ are closed, \( \cap_{\alpha \in A} E_{\alpha} \) is closed.

(b) If $E_j$, $1 \leq j \leq n$ are closed, \( \cup_{j=1}^{n} E_j \) is closed.


We have already seen that there exist sets ($\emptyset$ and $\mathbb{R}$) that are both open and closed. We have also seen that there exist sets (like $[0, 1)$) that are neither open nor closed. Other examples appear in the problems at the end of the chapter.
11.1.2 Accumulation points of sets

In the previous subsection, we defined a closed set to be a set whose complement is open. We come up with a second characterization involving the notion of an accumulation point of a set.

**Definition 11.1.3.** Let $A \subseteq \mathbb{R}$ and let $x$ be an element of $\mathbb{R}$ not necessarily in $A$. Then $x$ is an accumulation point of $A$ if, for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ contains an element of $A$ different from $x$.

The intuition is supposed to be that an accumulation point of a set needn’t be in the set, but it has lots of elements of the set near it. This intuition is strengthened by the following proposition characterizing accumulation points.

**Proposition 11.1.4.** Let $A \subseteq \mathbb{R}$ and let $x$ be an element of $\mathbb{R}$ not necessarily in $A$. Then $x$ is an accumulation point of $A$ if and only if for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ contains infinitely many elements of $A$.

**Proof.** Suppose first that $x$ has the property that, for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ contains infinitely many elements of $A$. It then clearly contains an element of $A$ different from $x$, and so $x$ is indeed an accumulation point.

Conversely, suppose $x$ is an accumulation point of $A$. Let $\varepsilon > 0$ be given. Then $(x - \varepsilon, x + \varepsilon)$ contains an element $a_1$ of $A$ such that $a_1 \neq x$. Now set $\varepsilon_1 = |x - a_1|$ and note that $0 < \varepsilon_1 < \varepsilon$. Consider $(x - \varepsilon_1, x + \varepsilon_1)$. This interval contains an element $a_2$ of $A$ different from $x$. Furthermore, $a_2 \neq a_1$ because $a_1$ is not an element of $(x - \varepsilon_1, x + \varepsilon_1)$. Proceed inductively; having selected $a_1, \ldots, a_n$ distinct elements of $A$ different from $x$, set $\varepsilon_n = |x - a_n|$. Note that $0 < \varepsilon_n < \varepsilon_{n-1} < \ldots < \varepsilon_1 < \varepsilon$. Then take $a_{n+1}$ to be any element of $(x - \varepsilon_n, x + \varepsilon_n)$ different from $x$. Because no $a_j$ is in $(x - \varepsilon_n, x + \varepsilon_n)$, $a_{n+1}$ is also unequal to $a_j$ for all $1 \leq j \leq n$. In this manner we obtain an infinite number of $a_n$ in $A$ in the interval $(x - \varepsilon, x + \varepsilon)$, as desired. \qed

**Exercise 11.1.9.** Find all accumulation points of the sets $A = \{ \frac{1}{n} : n \in \mathbb{N} \}$, $B = [-1, 1]$, and $C = (-1, 1)$.

The idea from the second half of the proof of the last proposition can be modified to establish connections between accumulation points of sets and limits of sequences in the set.

**Proposition 11.1.5.** If $x$ is an accumulation point of $A$, there exists a sequence \{a_n\} consisting of elements of $A$ such that $\lim_{n \to \infty} a_n = x$.

**Proof.** For each $n$, consider $(x - 1/n, x + 1/n)$. Because $x$ is an accumulation point of $A$, there exists $a_n \in (x - 1/n, x + 1/n) \cap A$. We claim \{a_n\} has limit $x$. Indeed, let $\varepsilon > 0$ be given. Then there exists $N$ such that $1/N < \varepsilon$. If $n \geq N$,

$$x - \varepsilon < x - \frac{1}{N} < x - \frac{1}{n} < a_n < x + \frac{1}{n} < x + \frac{1}{N} < x + \varepsilon,$$

i.e., if $n \geq N$, $|a_n - x| < \varepsilon$. \qed
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Exercise 11.1.10. Modify the proof of Proposition 11.1.5 to show that, if \( x \) is an accumulation point of \( A \), there exists a sequence \( \{a_n\} \) of distinct elements of \( A \) such that \( \lim_{n \to \infty} a_n = x \).

The next theorem is the main goal of this subsection.

Theorem 11.1.1. \( E \subseteq \mathbb{R} \) is closed if and only if \( E \) contains all its accumulation points.

Proof. Suppose first that \( E \) is closed, so that \( E^c \) is open. We prove that if \( x \) is an accumulation point of \( E \), then \( x \in E \) by proving the contrapositive, i.e., if \( x \) is not in \( E \), then \( x \) is not an accumulation point.

If \( x \notin E \), then \( x \in E^c \). Because \( E^c \) is open, there exists \( \varepsilon > 0 \) such that \( (x - \varepsilon, x + \varepsilon) \subseteq E^c \). This set thus contains no points of \( E \) and hence \( x \) is not an accumulation point.

Conversely, suppose \( E \) contains all its accumulation points. We must show \( E \) is closed, i.e., that \( E^c \) is open. Take \( x \in E^c \). Because \( E \) contains all its accumulation points, \( x \) is not an accumulation point of \( E \). Thus there exists \( \varepsilon > 0 \) such that \( (x - \varepsilon, x + \varepsilon) \) contains no element of \( E \) different from \( x \). Because \( x \) itself is not in \( E \), we have \( (x - \varepsilon, x + \varepsilon) \subseteq E^c \). Thus \( E^c \) is indeed open. \( \square \)

Exercise 11.1.11. Prove that \( \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{0\} \) is closed.

Exercise 11.1.12. Consider the finite set \( A = \{a_1, \ldots, a_n\} \). What are the accumulation points of \( A \)? Prove that \( A \) is closed by arguing that it contains all its accumulation points.

11.1.3 Compact sets

Consider again Theorem 11.0.1. In that theorem, the hypothesis on \( f \) is that it is continuous on \( [a, b] \). We now know that \( [a, b] \) is an example of a closed set. Is the theorem still true if we replace \( [a, b] \) by an arbitrary closed set? No. We know, for example, that \( \mathbb{R} \) is closed. The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^2 \) is an example of a continuous function that is unbounded on \( \mathbb{R} \), hence does not attain a maximum value. Thus requiring the domain to be closed is not enough; we have not yet found the right topological property of the domain. It turns out that we require the domain to be closed and bounded. The sets with both these properties are the compact sets, though we define them in a manner that initially makes it hard to see that they have these properties.

Before we give the definition of compact set, we must talk briefly about subsequences of sequences of real numbers.

Definition 11.1.4. Let \( \{a_n\} \) be a sequence of real numbers. \( \{b_k\} \) is a subsequence if there exists a strictly increasing function \( n : \mathbb{N}_0 \to \mathbb{N}_0 \) such that \( b_k = a_{n(k)} \) for all \( k \). We often write \( \{a_{n_k}\} \) for a subsequence instead of \( \{a_{n(k)}\} \).

As always, we look at an example to clarify the definition.
Example 11.1.2. Let \( a_n = \frac{1}{n+1} \). If we define \( n : \mathbb{N}_0 \to \mathbb{N}_0 \) by 

\[
n(k) = 2k,
\]

we obtain the subsequence 

\[
\{a_{n(0)}, a_{n(1)}, a_{n(2)}, \ldots, a_{n(k)}, \ldots\} = \{a_0, a_2, \ldots, a_{2k}, \ldots\} = \{\frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2k+1}, \ldots\}.
\]

If instead we take \( n(k) = k^2 \), we obtain the subsequence with \( b_k = \frac{1}{k^2+1} \). Recalling that sequences are themselves functions on \( \mathbb{N}_0 \), we observe that all we are really doing to get sequence \( b \) is taking \( a \) and composing it with a strictly increasing function \( n \) on \( \mathbb{N}_0 \) to obtain \( a \circ n : \mathbb{N}_0 \to \mathbb{R} \).

Exercise 11.1.13. Consider sequence \( a \) defined by \( a_n = (-1)^n \frac{n}{n+1} \). What sequence is obtained by taking \( n(k) = 2k \) or \( n(k) = 2k + 1 \)?

In the previous exercise, we have an example of a sequence that is divergent but for which there exist subsequences that are convergent. We are motivated to make a definition.

Definition 11.1.5. Let \( a = \{a_n\} \) be a sequence of real numbers. \( L \in \mathbb{R} \) is a limit point of \( a \) if there exists a subsequence of \( a \) with limit \( L \).

The sequence in Exercise 11.1.13 has limit points 1 and \(-1\). We consider two more examples, one elementary and one a little more interesting.

Example 11.1.3. Let \( a \) be the sequence with \( n \)-th term \( a_n = n \) mod 3. In other words,

\[
a = \{0, 1, 2, 0, 1, 2, 0, 1, 2, \ldots\}.
\]

This sequence has limit points 0, 1, and 2.

It is possible for a sequence to have infinitely many limit points. For example,

\[
a = \{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots\}
\]

has every natural number as a limit point.

We prove one result about limit points of convergent sequences.

Proposition 11.1.6. Let \( \{a_n\} \) be a convergent sequence with limit \( L \). Then every subsequence has limit \( L \). Consequently, a convergent sequence has a single limit point.

Proof. Let \( \{b_k\} = \{a_{n_k}\} \) be any subsequence and let \( \varepsilon > 0 \) be given. Because the full sequence \( \{a_n\} \) converges to \( L \), for this \( \varepsilon > 0 \), there exists \( N \) such that for all \( n \geq N \), \( |a_n - L| < \varepsilon \). Because \( k \mapsto n(k) \) is strictly increasing, if \( k \geq N \), \( n_k \geq N \) and so

\[
|b_k - L| = |a_{n_k} - L| < \varepsilon.
\]

Thus \( \{b_k\} \) has limit \( L \) as well. \( \square \)
We are now ready to define compactness.

**Definition 11.1.6.** A set $K \subseteq \mathbb{R}$ is **compact** if for every sequence $\{a_n\}$ with $a_n \in K$ for all $n$, $\{a_n\}$ has a limit point in $K$.

What kinds of sets are compact? To begin with, sets with only finitely many elements are compact.

**Proposition 11.1.7.** Let $K = \{x_1, \ldots, x_M\}$ be a finite subset of $\mathbb{R}$. Then $K$ is compact.

*Proof.* Let $a = \{a_n\}$ be a sequence such that $a_n \in K$ for all $n$. Because $K$ is finite, some element of $K$ must be equal to $a_n$ for infinitely many $n$. Let $x_j$ be such an element of $K$. Let $A = \{n \in \mathbb{N} : a_n = x_j\}$. The set $A$ is an infinite (hence non-empty) subset of $\mathbb{N}$. By the well-ordering principle for $\mathbb{N}$, $A$ has a smallest element $n_1$. Proceed inductively; having obtained $n_1 < n_2 < \ldots < n_k$, $k \geq 1$, we obtain $n_{k+1}$ as follows. The set $A \setminus \{n_1, \ldots, n_k\}$ is still an infinite (hence non-empty) subset of $\mathbb{N}$ and so it has a smallest element $n_{k+1}$. Clearly $n_{k+1} > n_k$ for otherwise, at step $k$, $n_k$ would not have been the smallest element of $A \setminus \{n_1, \ldots, n_{k-1}\}$.

Now define $b = \{b_k\}$ by setting $b_k = a_{n_k}$. Because $k \mapsto n_k$ is strictly increasing, $b$ is a subsequence of $a$. Because it is a constant sequence with every term equal to $x_j$, it is clearly convergent to the element $x_j$ of $K$. Thus $K$ is compact. \(\Box\)

**Exercise 11.1.14.** Negate the definition of compactness and use the negation to show that $\mathbb{Z}$ is not compact.

**Exercise 11.1.15.** Show that $(0, 1)$ is not compact.

Another canonical example of a compact set is a closed and bounded interval.

**Proposition 11.1.8.** Suppose $a < b$. Then $K = [a, b]$ is compact.

*Proof.* Let $\{x_n\}$ be a sequence in $[a, b]$. Set $c_1 = \frac{a + b}{2}$. One of the sets $[a, c_1]$ or $[c_1, b]$ contains infinitely many terms of the sequence $\{x_n\}$. Let $I_1 = [a_1, b_1]$ be one of the sets with this property and let $x_{n_1}$ be a term in the sequence $\{x_n\}$ in $I_1$. We proceed inductively to define the sets $I_k$ and the terms $x_{n_k}$; having obtained closed intervals $I_{k-1} \subseteq \ldots \subseteq I_1$ and terms $x_{n_1}, \ldots, x_{n_{k-1}}$ of $\{x_n\}$ with $n_1 < \ldots < n_{k-1}$, consider the midpoint $c_k = \frac{a_{k-1} + b_{k-1}}{2}$ of $I_{k-1} = [a_{k-1}, b_{k-1}]$. Because $I_{k-1}$ contains infinitely many terms of the sequence $\{x_n\}$, one of the subintervals $[a_{k-1}, c_k]$ or $[c_k, b_{k-1}]$ does as well. Let $I_k = [a_k, b_k]$ be one of these intervals having this property and let $x_{n_k}$ be a term in the sequence $\{x_n\}$ that is in $I_k$ and for which $n_k > n_{k-1}$.

In this manner we have constructed a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. We claim it is convergent to an element of $[a, b]$. We prove this claim by proving that the sequence $\{x_{n_k}\}$ is Cauchy. Let $\varepsilon > 0$ be given. We must show that there exists $J$ such that for all $j, k \geq J$, $|x_{n_j} - x_{n_k}| < \varepsilon$. Assume without loss of
generality that \( k > j \) and observe that \( I_k \subseteq I_j \) for all \( k \). Observe also that because \( b_j - a_j < \frac{1}{2^j}(b - a) \), if \( x_{n_k}, x_{n_j} \in I_j \),

\[
|x_{n_k} - x_{n_j}| \leq \frac{1}{2^j}(b - a).
\]

Now, \( \frac{1}{2^j}(b - a) < \varepsilon \) if \( j > \log_2 \left( \frac{b - a}{\varepsilon} \right) \). Thus if we take \( J \) to be any natural number larger that \( \log_2 \left( \frac{b - a}{\varepsilon} \right) \), then for all \( k > j \geq J \), \( |x_{n_k} - x_{n_j}| < \varepsilon \). We conclude that \( \{x_{n_k}\} \) is a subsequence of \( \{x_n\} \) that converges to some \( y \). Because for all \( k \geq J \), \( x_{n_k} \in I_J \), for all such \( k \),

\[
a_j \leq x_{n_k} \leq b_j.
\]

Because non-strict inequalities are preserved when taking limits, \( a_j \leq y \leq b_j \). In particular, \( y \in [a, b] \). Because \( \{x_n\} \) was an arbitrary sequence in \([a, b] \), we have shown that every sequence in \([a, b] \) has a limit point in \([a, b] \). Thus, by definition, \([a, b] \) is compact.

We prove a general theorem characterizing compact sets.

**Theorem 11.1.2.** \( K \subseteq \mathbb{R} \) is compact if and only if it is closed and bounded.

**Proof.** We begin by proving that if \( K \) is compact, then it is closed and bounded. We accomplish this goal by proving the contrapositive.

Thus we first suppose \( K \) is not bounded and show that \( K \) is not compact. Indeed, if \( K \) is not bounded, then given \( n \in \mathbb{N} \) there exists \( x_n \in K \) such that \( |x_n| > n \). Form the sequence \( \{x_n\} \). Because every subsequence of \( \{x_n\} \) is also clearly unbounded, \( \{x_n\} \) has no convergent subsequences, hence no limit points. Thus \( K \) is not compact.

Suppose next that \( K \) is not closed. Then by Theorem 11.1.1, there exists an accumulation point \( x \) of \( K \) such that \( x \notin K \). By Proposition 11.1.5, there exists a sequence \( \{a_n\} \) such that \( a_n \in K \) and \( \lim_{n \to \infty} a_n = x \). Because \( \{a_n\} \) is convergent, \( x \) is its only limit point. Thus this sequence is an example of a sequence in \( K \) with no limit point in \( K \). Thus \( K \) is not compact.

Next we must show that, if \( K \) is closed and bounded, then \( K \) is compact. The structure of this proof is virtually identical to that of our proof of Proposition 11.1.8. Because \( K \) is bounded, there exists \( M > 0 \) such that \( K \subseteq [-M, M] \). Let \( \{x_n\} \) be a sequence in \( K \). Set \( c_1 = \frac{-M + M}{2} = 0 \). Because \( K \) is contained in their union, one of the intervals \([-M, c_1]\) or \([c_1, M]\) contains infinitely many terms of the sequence \( \{x_n\} \). Let \( I_1 = [a_1, b_1] \) be one such interval with this property and let \( x_{n_1} \) be a term in the sequence \( \{x_n\} \) in \( I_1 \). We proceed inductively; having defined closed intervals \( I_{k-1} \subseteq \ldots \subseteq I_1 \) and terms \( x_{n_1}, \ldots, x_{n_{k-1}} \) of \( \{x_n\} \) with \( n_1 < \ldots < n_{k-1} \), consider the midpoint \( c_k = \frac{a_{k-1} + b_{k-1}}{2} \) of \( I_{k-1} = [a_{k-1}, b_{k-1}] \). Because \( I_{k-1} \) contains infinitely many terms of the sequence \( \{x_n\} \), one of the subintervals \([a_{k-1}, c_k]\) or \([c_k, b_{k-1}]\) does as well. Let \( I_k = [a_k, b_k] \) be one such interval and let \( x_{n_k} \) be a term in the sequence \( \{x_n\} \) that is in \( I_k \) and for which \( n_k > n_{k-1} \). We have constructed a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \). We claim it converges to an element of \( K \). We prove this claim by proving that \( \{x_{n_k}\} \) is
Cauchy. Let \( \varepsilon > 0 \) be given. We must show that there exists \( J \) such that for all \( j, k \geq J \), \( |x_{n_j} - x_{n_k}| < \varepsilon \). Assume without loss of generality that \( k > j \) and observe that \( I_k \subseteq I_j \) for all \( k \). Observe also that because \( b_j - a_j < \frac{M}{2^j} \), if \( x_{n_k}, x_{n_j} \in I_j \),
\[
|x_{n_k} - x_{n_j}| \leq \frac{M}{2^j - 1}.
\]
Now, \( \frac{M}{2^{j+1}} < \varepsilon \) if \( j > \log_2 \left( \frac{M}{\varepsilon} \right) + 1 \). Thus if we take \( J \) to be any natural number larger than \( \log_2 \left( \frac{M}{\varepsilon} \right) + 1 \), then for all \( k > j \geq J \), \( |x_{n_k} - x_{n_j}| < \varepsilon \). We conclude that \( \{x_{n_j}\} \) is a subsequence of \( \{x_n\} \) that converges to some \( y \). We claim that \( y \in K \). If not, \( y \in K^c \). Because \( K \) is closed, \( K^c \) is open, there exists \( \varepsilon > 0 \) such that \( (y - \varepsilon, y + \varepsilon) \) is contained in \( K^c \). This contradicts the fact that the \( x_n \) are in \( K \) and that \( \lim_{n \to \infty} x_n = y \). Thus \( y \in K \) and \( K \) is compact.

11.2 A First Glimpse at the Notion of Measure

11.2.1 Measuring open sets

11.2.2 Measure zero

11.2.3 The Cantor set

11.3 Problems

1. For each of the following, either give an example of such a set or explain why no such set exists.

   (a) An infinite collection of open sets \( U_j, j \in \mathbb{N} \) whose intersection is open.

   (b) A closed set with no accumulation points.

   (c) An open set with no accumulation points.

   (d) A closed set whose complement is also closed.

   (e) A subset of \([0, 1]\) with no accumulation points.

   (f) An infinite subset of \([0, 1]\) with no accumulation points.

2. In this problem we explore the topological properties of the set \( \mathbb{Q} \) of rational numbers.

   (a) Let \( x \) be a rational number. Show that for every \( \varepsilon > 0 \), \( (x - \varepsilon, x + \varepsilon) \) contains a rational number different from \( x \).

   (b) Let \( x \) be a rational number. Show that for every \( \varepsilon > 0 \), \( (x - \varepsilon, x + \varepsilon) \) contains an irrational number.

   (c) Now let \( x \) be an irrational number. Show that for any \( \varepsilon > 0 \), \( (x - \varepsilon, x + \varepsilon) \) contains both a rational number and an irrational number different from \( x \).
(d) Is $\mathbb{Q}$ open? Closed?
(e) Which elements of $\mathbb{R}$ are accumulation points of $\mathbb{Q}$?

3. For each of the following, give an example of a sequence with the stated property or explain why no such sequence exists.

(a) A sequence with three limit points.
(b) A divergent sequence with only one limit point.
(c) A convergent sequence with two distinct limit points.

4. For each of the following, either give an example of a subset of $\mathbb{R}$ with the stated property or explain why no such set exists.

(a) A closed set that is not compact.
(b) A compact set that is not closed.
(c) A bounded set that is not compact.
(d) A compact set with no accumulation points.