Chapter 10

Series, Part 1

10.1 Basic Notions

In standard English, the words sequence and series are sometimes used interchangeably. It would be just as correct to describe the sequence of steps one must follow to serve a tennis ball as it would be to describe the series of steps one must follow. In mathematics, on the other hand, a sequence and a series are not the same thing. Roughly speaking, a sequence is an ordered list whereas a series is the (formal) sum of terms in a sequence. For example,

\[1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\]

is a sequence, but

\[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\]

is a series. The reason we have said that a series is the formal sum of terms in a sequence is that we have not yet said what it means to add up infinitely many things. Sometimes it will make sense to add up infinitely many things. For example, we will see that it makes good sense to say that the series above sums to, or converges to 2. On the other hand, in many situations it will not make sense to add up infinitely many numbers. For example, we will see that the series

\[1 - 1 + 1 - 1 + 1 - 1 + \ldots\]

doesn’t converge.

Of course, we need a precise definition.

Definition 10.1.1. Let \(\{a_n\}\) be sequence of real numbers. The series \(\sum_{n=0}^{\infty} a_n\) is the formal sum of the terms of the sequence. For each \(N \in \mathbb{N}_0\), we define the \(N\)-th partial sum of the series by \(A_N = \sum_{n=0}^{N} a_n\). We say the series \(\sum_{n=0}^{\infty} a_n\) converges if the sequence \(\{A_N\}\) has a limit \(L\), and we write \(\sum_{n=0}^{\infty} a_n = L\) in this case. Otherwise we say the series diverges.
Let’s begin our exploration of this definition with the two series above. Consider first
\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} \frac{1}{2^n}. \tag{10.1.1} \]

Does this series converge? Our definition requires us to consider the corresponding sequence \( \{A_N\} \) of partial sums. Let’s try to understand this sequence. We begin by generating the first few terms.

\[
A_0 = \sum_{n=0}^{0} \frac{1}{2^n} = 1 \\
A_1 = \sum_{n=0}^{1} \frac{1}{2^n} = 1 + \frac{1}{2} = \frac{3}{2} \\
A_2 = \sum_{n=0}^{2} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} \\
A_3 = \sum_{n=0}^{3} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}.
\]

We conjecture that
\[ A_N = \sum_{n=0}^{N} \frac{1}{2^n} = \frac{2^{N+1} - 1}{2^N} = 2 - \frac{1}{2^N}. \tag{10.1.2} \]

This conjecture is, in fact, true. We leave the easy proof to the next exercise.

**Exercise 10.1.1.** Prove (10.1.2). Several methods are possible; one method is to use induction on \( N \).

We may now determine whether our series converges. The definition says the series (10.1.1) converges to the number \( L \) if \( \lim_{N \to \infty} A_N \) exists and equals \( L \). Because
\[
\lim_{N \to \infty} A_N = \lim_{N \to \infty} \left( 2 - \frac{1}{2^N} \right) = 2,
\]
we conclude that our original series converges to 2.

Next we consider the series
\[ 1 - 1 + 1 - 1 + \ldots = \sum_{n=0}^{\infty} (-1)^n. \tag{10.1.3} \]
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\[ A_0 = \sum_{n=0}^{0} (-1)^n = 1 \]

\[ A_1 = \sum_{n=0}^{1} (-1)^n = 1 - 1 = 0 \]

\[ A_2 = \sum_{n=0}^{2} (-1)^n = 1 - 1 + 1 = 1 \]

\[ A_3 = \sum_{n=0}^{3} (-1)^n = 1 - 1 + 1 - 1 = 0. \]

We easily see (by an induction proof that we can do in our heads at this point) that

\[ A_N = \begin{cases} 1 & \text{if } N \text{ is even} \\ 0 & \text{if } N \text{ is odd}. \end{cases} \]

Thus the sequence \( \{A_N\} \) does not have a limit. Because the sequence of partial sums for the series \( \sum_{n=0}^{\infty} (-1)^n \) does not have a limit, the series does not converge.

10.1.1 Exploring the sequence of partial sums graphically and numerically

Because the two series considered in the previous subsection were quite simple, we could easily write down a formula for the \( N \)-th term of the sequence of partial sums and see if the sequence of partial sums had a limit. In many cases, writing down such a formula is difficult or impossible. We may still want to have a feeling for the behavior of the partial sums before we try to prove that a series does or does not converge. In such cases, we can explore the partial sums graphically and numerically.

Example 10.1.1. Consider the series

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \]

In order to determine whether the series converges or diverges, we must examine the associated sequence of partial sums. By definition,

\[ S_0 = \sum_{n=0}^{0} \frac{(-1)^n}{(n+1)^2} = 1 \]

\[ S_1 = \sum_{n=0}^{1} \frac{(-1)^n}{(n+1)^2} = 1 + \frac{(-1)^1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4} \]

\[ S_2 = \sum_{n=0}^{2} \frac{(-1)^n}{(n+1)^2} = 1 - \frac{1}{4} + \frac{1}{9} = \frac{31}{36}. \]
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Generating partial sums by hand quickly becomes tedious. We thus instead write a few lines of Python code to generate and plot the first 50 partial sums.

```python
# The first partial sum is 1.
partialSums = [1]

# Each time through the loop, add to the previous partial sum
# the next term in the series
for N in range(1, 50):
    partialSums.append(partialSums[N-1] + (-1.0)**(N) / ((N+1)**2))

print(partialSums)
plot(partialSums, ".")
show()
```

Here's the plot:
The plot certainly suggests that the sequence of partial sums is convergent. In fact it looks as if the partial sums with even index form a decreasing sequence that is bounded below and the partial sums with odd index form an increasing sequence that is bounded above. Furthermore, these sequences appear to have a common limit. From the numerical output, we see that the last three partial sums are

$$0.822254538698166, \ 0.8226710318260294, \ 0.822710318260295,$$

suggesting that the partial sums approach some real number near 0.822.

Of course this numerical and graphical evidence simply leads us to conjecture that the series $\sum_{n=0}^{\infty} \frac{n}{n+1}$ converges. We still require a proof. Such a proof will follow; at this point, our aims are simply to understand clearly the definition of the convergence of a series and to explore examples.

**Example 10.1.2.** Next we consider $\sum_{n=0}^{\infty} \frac{n}{n+1}$. Observe,

$$A_1 = \frac{1}{2},$$
$$A_2 = \frac{1}{2} + \frac{2}{3},$$
$$A_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4},$$
$$A_4 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}.$$

Notice that to go from the $A_N$ to $A_{N+1}$, we must add $\frac{N+1}{N+2}$, which is very close to 1 for large $N$. Thus it is reasonable to guess that the sequence $\{A_N\}$ is unbounded, hence divergent.

We prove this claim. Let $K > 0$ be given. We must find $N$ so that $A_N > K$. We can obtain a very simple lower bound for $A_N$; observe,

$$A_2 = \frac{1}{2} + \frac{2}{3} > \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1,$$
$$A_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2},$$

and so on. An easy induction argument shows that, for all $n > 2$,

$$A_N > \frac{N}{2}.$$

Because this is greater than $K$ if $N > 2K$, our claim follows. Because the sequence $\{A_N\}$ is divergent, the series $\sum n = 0^\infty \frac{n}{n+1}$ is divergent.

**Exercise 10.1.2.** Write the code to generate and plot the first 50 partial sums of the series $\sum_{n=0}^{\infty} \frac{n}{n+1}$.
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10.1.2 Basic properties of convergent series

The examples we have considered are of the form $\sum_{n=0}^{\infty} a_n$ for (a) $a_n = 2^{-n}$, (b) $a_n = (-1)^n$, (c) $a_n = (-1)^n/(n+1)^2$, and (d) $a_n = n/(n+1)$. You may have noticed that in the case of the two divergent series (associated with (b) and (d)), the terms $a_n$ themselves do not have limit zero, whereas in the case of the two convergent series (associated with (a) and (c)), the terms $a_n$ have limit zero. We will see shortly that the question of convergence of a series cannot be so easily settled; there are many divergent series $\sum a_n$ for which $\lim_{n \to \infty} a_n = 0$. We have, however, discovered a very important necessary condition.

**Proposition 10.1.1.** Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers. If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

**Proof.** Let $A_N = \sum_{n=0}^{N} a_n$ be the $N$-th partial sum of the series. By hypothesis, $\{A_N\}$ is a convergent sequence of real numbers; denote its limit by $L$. Observe that, for $N \geq 1$,

$$a_N = A_N - A_{N-1}. \quad (10.1.5)$$

Because $\lim_{N \to \infty} A_N = L$ and $\lim_{N \to \infty} A_{N-1} = L$, the limit of the right-hand-side of (10.1.5) exists. Thus

$$\lim_{N \to \infty} a_N = \lim_{N \to \infty} (A_N - A_{N-1}) = \lim_{N \to \infty} A_N - \lim_{N \to \infty} A_{N-1} = L - L = 0,$$

as claimed. \qed

Such a simple necessary condition for convergence gives us a very easy way to show that certain series are divergent; If the sequence $\{a_n\}$ does not have limit 0, the series $\sum_{n=0}^{\infty} a_n$ is divergent.

**Exercise 10.1.3.** Consider again the series $\sum_{n=0}^{\infty} \frac{n}{n+1}$. In light of the above discussion, give a second proof of its divergence.

We close this section with an easy proposition about convergent series whose proof is left as an exercise.

**Proposition 10.1.2.** Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent. Let $c \in \mathbb{R}$.

1. $\sum_{n=0}^{\infty} ca_n$ is convergent with sum $c \sum_{n=0}^{\infty} a_n$.

2. $\sum_{n=0}^{\infty} (a_n + b_n)$ is convergent with sum $\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$.

**Exercise 10.1.4.** Prove Proposition 10.1.2.

10.1.3 Series that diverge slowly: the harmonic series

If the condition in Proposition 10.1.1 (that $\lim_{n \to \infty} a_n = 0$) were both necessary and sufficient for the convergence of $\sum a_n$, the study of infinite series would be comparatively simple. Alas, the condition is not sufficient. To show this, we
must give an example of a series whose terms have limit zero but which is nonetheless divergent. The canonical example is the series whose terms are the reciprocals of the natural numbers:

\[ \sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \] (10.1.6)

The series (10.1.6) is called the harmonic series.

Of course, to determine whether the series converges, we must consider the sequence \( \{A_N\} \) of partial sums. Because all of the terms of the series are positive, \( A_0 < A_1 < A_2 < \ldots \), i.e., the sequence of partial sums is increasing. We know from our study of sequences that an increasing sequence of real numbers has a real limit if and only if it is bounded above. Thus our task is to determine whether or not the sequence \( \{A_N\} \) is bounded.

So we proceed as above, generating and plotting the first 50 partial sums:
Is the sequence of partial sums bounded? It’s not clear. Let’s look at the first 500,
and the first 5000 partial sums.
We find that

\[ A_{49} = 4.499205338329423, \quad A_{499} = 6.79282342999052, \quad A_{4999} = 9.094508852984404. \]

Although initially these data may not seem to shed much light on the situation, they actually contain a clue; the function \( N \mapsto A_N \) seems to have the property that multiplying the input by 10 (i.e., considering the 50-th term, then the 500-th, then the 5000-th) seems to add a nearly constant amount of about 2.3 to the function. We know from our pre-calculus days that logarithmic functions are defined by this property. We thus suspect that \( A_N \) is approximately logarithmic in \( N \). Furthermore, because logarithmic functions are unbounded, we suspect that the same might be true of the sequence \( \{A_N\} \). If we can prove this, we will be able to conclude that the harmonic series is in fact divergent.

**Exercise 10.1.5.** Write the Python code necessary to generate the plots above and the numerical values of \( A_{49}, A_{499}, A_{4999}, \) and \( A_{49999} \).

We therefore prove that the sequence \( \{A_N\} \) is unbounded. Thus let \( K > 0 \) be given. We must show that there exists \( M \) such that, if \( N \geq M, A_N > K \). Observe,

\[
A_0 = 1 = \frac{2}{2}, \\
A_1 = 1 + \frac{1}{2} = \frac{3}{2}, \\
A_3 = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) \geq \frac{3}{2} + \frac{1}{2} = \frac{4}{2}, \\
A_7 = A_3 + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) \geq \frac{4}{2} + \frac{4}{8} = \frac{5}{2}.
\]

A simple induction argument shows that

\[
A_{2^n-1} \geq \frac{n+2}{2}.
\]

Note that

\[
\frac{n+2}{2} > K \iff n > 2K - 2.
\]

Set \( n_0 = 2\lceil K \rceil - 1 \). Then if \( N = 2^{n_0} - 1, \)

\[
A_N = A_{2^{n_0}-1} > K.
\]

We have thus shown that the sequence \( \{A_N\} \) of partial sums of the harmonic series is unbounded, hence divergent. We conclude that the harmonic series is divergent.

**Exercise 10.1.6.** Let \( A_N \) denote the \( N \)-th partial sum of the harmonic series, as above. How large must \( N \) be in order to have \( A_N > 10 \) ? \( A_N > 100 \) ?
10.2 Infinite geometric series

Recall that a sequence is geometric if \( g_n = ar^n \) for \( a, r \) real constants and \( n \in \mathbb{N}_0 \). We call \( r \) the common ratio. An infinite geometric series is thus the series \( \sum_{n=0}^{\infty} ar^n \). For which \( a \) and \( r \) is this series convergent?

We consider the sequence of partial sums

\[
S_N = \sum_{n=0}^{N} ar^n.
\]

In Problem ?? we sought a simple formula for this finite geometric series. The result for \( r \neq 1 \) is

\[
S_N = \frac{a(1 - r^{N+1})}{1 - r}.
\]

This sequence has a limit if and only if \( \lim_{N \to \infty} r^{N+1} \) has a limit. One easily shows that \( \lim_{N \to \infty} r^{N+1} = 0 \) if \( |r| < 1 \) and diverges if \( r > 1 \) or \( r \leq -1 \). Thus \( S_N \) converges if and only if \( |r| < 1 \), in which case its limit is \( \frac{a}{1 - r} \). Observe that if \( r = 1 \), \( S_N = a(N + 1) \) which diverges unless \( a = 0 \). We have therefore proved:

**Proposition 10.2.1.** Consider the infinite geometric series \( \sum_{n=0}^{\infty} ar^n \) for \( a \neq 0 \).

Then

(a) if \( |r| < 1 \), the series converges to \( \frac{a}{1 - r} \).

(b) if \( |r| \geq 1 \), the series diverges.

**Exercise 10.2.1.** Each of the following series is geometric. Determine whether each is convergent or divergent. Find the sum of each convergent series.

(a) \( \sum_{n=0}^{\infty} 3^{-n} \).

(b) \( \sum_{n=1}^{\infty} \frac{(-2)^n}{5^n+1} \).

(c) \( \sum_{n=1}^{\infty} \frac{4^{n-1}}{3^n} \).

(d) \( \sum_{n=0}^{\infty} x^{2n} \) for \( |x| < 1 \).
10.3 Tests for Convergence of Series

At this point, we have a good understanding of the definition of convergence for a series and a complete understanding of infinite geometric series. Our goal now is to develop theorems that can help us settle the question of convergence for much more general series.

**Proposition 10.3.1.** Let \( \{a_n\} \) be a sequence of real numbers. If \( \sum |a_n| \) converges, then \( \sum a_n \) converges.

**Proof.** Let \( A_N = \sum_{n=0}^{N} a_n \). We must show that \( \{A_N\} \) converges. We will prove this assertion by showing that \( \{A_N\} \) is Cauchy.

Let \( \varepsilon > 0 \) be given. We must show that there exists \( K \) such that, for all \( N, M \geq K \), \( |A_N - A_M| < \varepsilon \). Assume without loss of generality that \( N > M \).

Let \( S_N = \sum_{n=0}^{N} |a_n| \). Observe,

\[
|A_N - A_M| = \left| \sum_{n=0}^{N} a_n - \sum_{n=0}^{M} a_n \right| = \left| \sum_{n=M+1}^{N} a_n \right| = \sum_{n=M+1}^{N} |a_n|, \quad \text{(by the triangle inequality)}
\]

Because \( \sum |a_n| \) converges, the sequence \( \{S_N\} \) is Cauchy. Thus for \( \varepsilon \) as above, there exists \( K \) such that, for all \( N, M \geq K \), \( |S_N - S_M| < \varepsilon \). Thus for this same \( K \), if \( N > M \geq K \), \( |A_N - A_M| < \varepsilon \).

\[\Box\]

**Proposition 10.3.2** (Comparison Test). Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of non-negative real numbers. Suppose \( 0 \leq a_n \leq b_n \).

(a) If \( \sum b_n \) converges, then \( \sum a_n \) converges.

(b) If \( \sum a_n \) diverges, then \( \sum b_n \) diverges.

**Proof.** The proof of (a) is almost identical to the proof of Proposition 10.3.1 and is left to the reader as an exercise.

Consider (b). Let \( A_N = \sum_{n=0}^{N} a_n \) and \( B_N = \sum_{n=0}^{N} b_n \). Because the \( a_n \) are nonnegative, the sequence \( \{A_n\} \) is nondecreasing. If it were bounded above, it would be convergent by the Monotone Sequence Theorem. But the divergence of \( \sum a_n \) means that \( \{A_N\} \) is also divergent, hence unbounded in this case. We claim that \( \{B_N\} \) is thus also unbounded. Indeed, let \( M \in \mathbb{R} \) be given. Because
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\{A_N\} is unbounded, there exists \( N \) such that \( A_N \geq M \). But

\[
B_N = \sum_{n=0}^{N} b_n
\geq \sum_{n=0}^{N} a_n
= A_N
\geq M.
\]

We have established the claim. \( \square \)

**Exercise 10.3.1.** Prove part (a) of Proposition 10.3.2

**Exercise 10.3.2.** Prove that the conclusions of Proposition 10.3.2 still hold if we replace the hypothesis \( a_n \leq b_n \) with the hypothesis that there exists \( n_0 \) such that, for all \( n \geq n_0 \), \( a_n \leq b_n \).

**Exercise 10.3.3.** Consider \( \sum_{n=0}^{\infty} \frac{1}{n^p} \). Prove that this series is convergent by comparing it to a convergent geometric series.

In order to use the comparison test, we need a larger class of series whose convergence or divergence we can establish directly.

**Theorem 10.3.1 (p-series test).** Let \( 0 < p \) and consider the p-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) (or, with zero-indexing, \( \sum_{n=0}^{\infty} \frac{1}{(n+1)^p} \)). This series converges if \( p > 1 \) and diverges if \( 0 < p \leq 1 \).

This proof will resemble our proof that the harmonic series is divergent in which we obtained a lower bound for partial sums involving a number of terms that is a power of two. Thus we need a lemma estimating such partial sums for p-series.

**Lemma 10.3.1.** Let \( A_N = \sum_{n=1}^{N} \frac{1}{n^p} \). Then

\[
1 + \sum_{n=1}^{N} \frac{1}{2} (2^{1-p})^n \leq A_{2N},
\]

and

\[
A_{2N-1} \leq 1 + \sum_{n=1}^{N-1} (2^{1-p})^n.
\]

**Proof.** Each inequality is proved by induction on \( N \).

Consider first (10.3.1). When \( N = 1 \),

\[
A_{2^1} = 1 + \frac{1}{2^{p}}
= 1 + \frac{1}{2} \cdot 2^{1-p}
\]
and the result holds.

Now suppose the result holds for \( N \geq 1 \) and consider \( N + 1 \).

\[
A_{2^{N+1}} = \sum_{n=1}^{2^{N}} \frac{1}{n^p} + \sum_{n=2^{N+1}}^{2^{N+1}} \frac{1}{n^p}
\]

\[
\geq 1 + \sum_{n=1}^{N} \frac{1}{2} (2^{1-p})^n + \sum_{n=2^{N+1}}^{2^{N+1}} \frac{1}{(2^{N+1})^p}
\]

\[
= 1 + \sum_{n=1}^{N} \frac{1}{2} (2^{1-p})^n + \frac{1}{2} \cdot \frac{2^{N+1}}{(2^{N+1})^p}
\]

\[
\geq 1 + \sum_{n=1}^{N} \frac{1}{2} (2^{1-p})^n + \frac{1}{2} (2^{1-p})^{N+1}
\]

\[
\geq 1 + \sum_{n=1}^{N+1} \frac{1}{2} (2^{1-p})^n
\]

Thus the inequality holds for \( N + 1 \) and hence, by the principle of mathematical induction, for all \( N \).

The proof of the second inequality is also by induction. When \( N = 1 \),

\[
A_{2^1-1} = A_1
\]

\[
= 1
\]

\[
= 1 + \sum_{n=1}^{0} (2^{1-p})^n,
\]

because the sum is empty and equals 0. Thus the result holds for \( N = 1 \).

Suppose now the result holds for \( N \geq 1 \) and consider \( N + 1 \).

\[
A_{2^{N+1}-1} = \sum_{n=1}^{2^{N}-1} \frac{1}{n^p} + \sum_{n=2^{N}}^{2^{N+1}-1} \frac{1}{n^p}
\]

\[
\leq 1 + \sum_{n=1}^{N-1} (2^{1-p})^n + \sum_{n=2^{N}}^{2^{N+1}-1} \frac{1}{(2^N)^p}
\]

\[
= 1 + \sum_{n=1}^{N-1} (2^{1-p})^n + \frac{2^N}{(2^N)^p}
\]

\[
\leq 1 + \sum_{n=1}^{N-1} (2^{1-p})^n + (2^{1-p})^N
\]

\[
\leq 1 + \sum_{n=1}^{N} (2^{1-p})^n,
\]

and the result holds for \( N + 1 \). By the principle of mathematical induction, it holds for all \( N \). \( \square \)
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We may now prove our theorem on $p$-series.

Proof. We prove first that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$. Let $A_N = \sum_{n=1}^{N} \frac{1}{n^p}$. Clearly $\{A_N\}$ is non-decreasing, so it suffices to show that $\{A_N\}$ is bounded. Indeed, for any $N, N \leq 2^N - 1$, and so by the previous lemma,

$$A_N \leq A_{2^N-1} \leq 1 + \sum_{n=1}^{N-1} (2^{1-p})^n.$$  

The sum on the right is a partial sum of a geometric series with $r = 2^{1-p}$ and so is convergent if $|r| < 1$. Because $p > 1$, $2^{1} < 2^p$, or, equivalently, $2^{1-p} < 1$. We have therefore shown that, for all $N$,

$$A_N \leq 1 + \sum_{n=1}^{\infty} (2^{1-p})^n = 1 + \frac{2^{1-p}}{1 - 2^{1-p}}.$$  

Thus by the Monotone Sequence Theorem, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

We can approach the proof of the divergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ in several ways. One approach is to prove that $A_N$ is unbounded by using the lower bound for $A_{2^N}$ obtained in the previous lemma. A second proof (which is the one we give here) simply uses the Comparison Test and the result we have already proved that the $p$-series for $p = 1$ is divergent. Indeed, note that, for $p < 1$, for $n \geq 1$, $n^{1-p} \geq 1$, or, equivalently,

$$\frac{1}{n^p} \geq \frac{1}{n}.$$  

Thus because $\sum \frac{1}{n}$ is divergent, by the comparison test, $\sum \frac{1}{n^p}$ is divergent if $p < 1$.

The $p$-series test and the comparison test together allow us to determine the convergence or divergence of any series whose $n$-th term is a rational function of $n$. The next exercise asks you to consider three concrete examples; the problems at the end of the chapter ask you to formulate and prove a general statement.

Exercise 10.3.4. For each of the series below, use the comparison test to prove the convergence or divergence of the given series.

(a) $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$

(b) $\sum_{n=2}^{\infty} \frac{n}{n^2 - 1}$

(c) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + 1}}$

Exercise 10.3.5. Give a proof that $\sum \frac{1}{n^p}$ diverges for $p < 1$ using the lower bound (10.3.1) for $A_{2^N}$. 
Of course, we can compare a series with non-negative terms to a $p$-series even if its $n$-th term is not a rational function of $n$; all we need do is establish an inequality.

**Example 10.3.1.** Consider again $\sum_{n=0}^{\infty} \frac{1}{n!}$. Observe that for all $n \geq 2$,

$$n! \geq n(n-1) = n^2 - n \geq n^2 - \frac{1}{2} n^2 = \frac{1}{2} n^2$$

Thus for all $n \geq 2$,

$$\frac{1}{n!} \leq \frac{2}{n^2}.$$  

The series $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is convergent because it is a constant times the convergent $p$-series with $p = 2$. Thus by the comparison test (with the strengthening given in Exercise 10.3.2), $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent.

Several very useful tests for convergence come about by comparing a given series to a geometric series.

**Theorem 10.3.2 (Ratio Test).** Let $\{a_n\}$ be a sequence of nonzero real numbers.

Suppose $\lim_{n \to \infty} \left| \frac{a_n}{a_{n-1}} \right|$ exists and equals $L$. Then

(a) If $L < 1$, $\sum a_n$ converges.

(b) If $L > 1$, $\sum a_n$ diverges.

Note that this theorem says nothing about what happens when $L = 1$. We will see that there are examples of both convergent and divergent series for which $L = 1$.

**Proof.** Suppose first that $\lim_{n \to \infty} \left| \frac{a_n}{a_{n-1}} \right| = L < 1$. Let $\varepsilon$ be a strictly positive real number for which $L < L + \varepsilon < 1$. (Such an $\varepsilon$ exists; take, for example, $\varepsilon = (1 - L)/2$.) Because $\lim_{n \to \infty} \left| \frac{a_n}{a_{n-1}} \right| = L$, for this $\varepsilon$, there exists $N$ such that, for all $n \geq N$,

$$\left| \frac{a_n}{a_{n-1}} - L \right| < \varepsilon.$$  

This implies that, for all $n > N$,

$$\left| \frac{a_n}{a_{n-1}} \right| < L + \varepsilon \quad \iff \quad |a_n| < |a_{n-1}|(L + \varepsilon). \quad (10.3.3)$$

Iterating this inequality yields that, for all $n > N$,

$$|a_n| < |a_N|(L + \varepsilon)^{n-N} = \frac{|a_N|}{(L + \varepsilon)^N} \cdot (L + \varepsilon)^n. \quad (10.3.4)$$

The expression on the far right is the $n$-th term of a geometric series with common ratio $r = L + \varepsilon$. Because $|r| < 1$, this geometric series is convergent.
10.4. REPRESENTATIONS OF REAL NUMBERS

Thus by the comparison test (and Exercise 10.3.2), \( \sum |a_n| \) is convergent. Then by Proposition 10.3.1, \( \sum a_n \) is convergent as well. We have thus established (a).

Suppose next that \( \lim_{n \to \infty} \left| \frac{a_n}{a_{n-1}} \right| = L > 1 \) and let \( \varepsilon > 0 \) be such that \( L - \varepsilon > 1 \).

Because \( \lim_{n \to \infty} \left| \frac{a_n}{a_{n-1}} \right| = L \), for this \( \varepsilon \), there exists \( N \) such that, for all \( n \geq N \),

\[
\left| \frac{a_n}{a_{n-1}} - L \right| < \varepsilon.
\]

This implies that, for all \( n > N \),

\[
L - \varepsilon < \left| \frac{a_n}{a_{n-1}} \right| \iff |a_{n-1}|(L - \varepsilon) < |a_n|.
\]

Iterating gives, for all \( n > N \),

\[
|a_n| > \frac{|a_N|}{(L - \varepsilon)^N} \cdot (L - \varepsilon)^n.
\]

Because \( L - \varepsilon > 1 \), the terms \( a_n \) do not have limit 0 (in fact the sequence \( \{a_n\} \) is unbounded). Thus by Proposition 10.1.1, \( \sum a_n \) is divergent.

**Exercise 10.3.6.** Give another proof of the convergence of \( \sum_{n=0}^{\infty} \frac{1}{n!} \), this time using the ratio test.

The ratio test is a favorite test for convergence because it is so easy to apply. Unfortunately it gives no information when \( L = 1 \), as the following exercise illustrates.

**Exercise 10.3.7.** Let \( a_n = \frac{1}{n} \) and \( b_n = \frac{1}{n^2} \). Show that \( \lim_{n \to \infty} \left| \frac{a_n}{a_{n-1}} \right| = 1 \) and \( \lim_{n \to \infty} \left| \frac{b_n}{b_{n-1}} \right| = 1 \). Because \( \sum a_n \) is divergent and \( \sum b_n \) is convergent, we conclude that the ratio test is inconclusive when \( L = 1 \).

In fact, the ratio test is always inconclusive when the \( n \)-th term is a rational function of \( n \).

**Exercise 10.3.8.** Let \( p \) and \( q \) be polynomials that are not the zero polynomial. Consider \( \sum_{n=n_0}^{\infty} \frac{p(n)}{q(n)} \) where \( n_0 \) is large enough that neither \( p(n) \) nor \( q(n) \) is zero for \( n \geq n_0 \). Show that the ratio test in inconclusive (i.e., that \( L = 1 \)).

### 10.4 Representations of real numbers

The topic of infinite series may seem esoteric, but you already have considerable informal familiarity with them; every time you describe a real number by giving its decimal representation, you are specifying a convergent series. In this section we explore decimal representations more closely, examine decimal representations of rational numbers, and discuss how to represent real numbers with respect to bases other than 10.
10.4.1 Base 10 representation

Let $0 \leq x < 1$. We consider it a familiar fact that we can write

$$x = 0.a_1a_2a_3a_4 \ldots$$

where each $a_n$ is one of the digits 0 through 9. What does such an expression really mean? It is a shorthand notation for the infinite series

$$\frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + \frac{a_4}{10000} + \ldots = \sum_{n=1}^{\infty} a_n 10^{-n}. \quad (10.4.1)$$

Two obvious questions present themselves:

1. Does every series of the form (10.4.1) converge to a real number $x$ satisfying $0 \leq x < 1$?

2. Can every real number $x$ satisfying $0 \leq x < 1$ be written in the form (10.4.1)?

The first proposition answers the first question essentially in the affirmative. (It is possible for the series (10.4.1) to converge to 1).

**Proposition 10.4.1.** Consider the infinite series $\sum_{n=1}^{\infty} a_n 10^{-n}$ where each $a_n$ is an integer satisfying $0 \leq a_n \leq 9$. This series converges to a real number $x$ satisfying $0 \leq x \leq 1$.

**Proof.** Let $x_N = \sum_{n=1}^{N} a_n 10^{-n}$. Because $\{x_N\}$ is non-decreasing, it is convergent if it is bounded. Observe,

$$x_N = \sum_{n=1}^{N} a_n 10^{-n} \leq \sum_{n=1}^{N} 9 \cdot 10^{-n} = 9 \sum_{n=1}^{N} 10^{-n} \leq 9 \sum_{n=1}^{\infty} 10^{-n} = \frac{9}{\frac{1}{10} - \frac{1}{10}} = 1.$$

In the above string of inequalities we have used the fact that $\sum_{n=1}^{N} 10^{-n}$ is a partial sum for a convergent geometric series with positive terms, hence is bounded above by the corresponding infinite geometric series.

Because $\{x_N\}$ is bounded and non-decreasing, there exists $x \in \mathbb{R}$ such that $\lim_{N \to \infty} x_N = x$. Because $0 \leq x_N \leq 1$ and non-strict inequalities are preserved when taking limits, $0 \leq x \leq 1$. \qed
In order to answer the second question above, we need to describe how to find the $a_n$ in (10.4.1) given $0 \leq x < 1$. For each such $x$ and each $N \in \mathbb{N}$, let $y_N$ be the largest element of $\{n \mid n \leq x\}$ for which $y_N \leq 10^N x$. Then set $x_N$ equal to $y_N \cdot 10^{-N}$. The numbers $x_N$ give us $N$-digit decimal approximations to $x$. Let us clarify with an example.

**Example 10.4.1.** Consider $x = 3/11$. We find $y_N$ and $x_N$ for $N = 1, 2, 3, 4$. 

- **$N = 1$**: Because $2 \leq 10 \cdot \frac{3}{11}$ but $3 > 10 \cdot \frac{3}{11}$, $y_1 = 2$ and $x_1 = \frac{y_1}{10} = \frac{2}{10} = 0.2$.
- **$N = 2$**: Because $27 \leq 100 \cdot \frac{3}{11}$ but $28 > 100 \cdot \frac{3}{11}$, $y_2 = 27$ and $x_2 = \frac{y_2}{100} = \frac{27}{100} = 0.27$.
- **$N = 3$**: Because $272 \leq 1000 \cdot \frac{3}{11}$ but $273 > 1000 \cdot \frac{3}{11}$, $y_3 = 272$ and $x_3 = \frac{y_3}{1000} = \frac{272}{1000} = 0.272$.
- **$N = 4$**: Because $2727 \leq 10000 \cdot \frac{3}{11}$ but $2728 > 10000 \cdot \frac{3}{11}$, $y_4 = 2727$ and $x_4 = \frac{y_4}{10000} = \frac{2727}{10000} = 0.2727$.

We then set $a_n = 10^n(x_n - x_{n-1})$, where we take $y_0 = x_0 = 0$. Observe that

$$a_n = 10^n(x_n - x_{n-1}) = 10^n \left( \frac{y_n}{10^n} - \frac{y_{n-1}}{10^{n-1}} \right) = y_n - 10y_{n-1}.$$ 

Because $y_n$ and $y_{n-1}$ are integers, $a_n$ is an integer. Also, because $y_{n-1} \leq 10^{n-1}x$, $10y_{n-1} \leq 10^n x$. Because $y_n$ is the largest element of $\{n \mid n \leq x\}$ with this property, $y_n \geq 10y_{n-1}$, so $a_n \geq 0$. Finally, because $y_{n-1} + 1 > 10^{n-1}x$,

$$a_n < 10^nx - 10(10^{n-1}x - 1) = 10.$$ 

Thus $a_n$ is indeed one of the digits 0 through 9. Because

$$x_N = \sum_{n=1}^{N} (x_n - x_{n-1}) = \sum_{n=1}^{N} \frac{a_n}{10^n},$$

the $a_n$ really are the digits in the decimal expansion of $x_N$.

We will have finished the proof that $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = x$ if we show that $\lim_{N \to \infty} x_N = x$. We already know that $\{x_N\}$ is non-decreasing because $x_N - x_{N-1} \geq 0$. We also know that $x_N \leq x$ for all $x$. Thus $\{x_N\}$ converges to some $L$. Because

$$y_N \leq 10^N x < y_N + 1,$$

we have

$$x_N \leq x < x_N + 10^{-N}.$$ 

Because non-strict inequalities are preserved when taking limits,

$$L \leq x \leq L$$

and, indeed, $L = x$. We have thus not only proved that every real number $0 \leq x < 1$ has a decimal representation, we have also shown how to obtain one.
10.4.2 Base 10 representation of rational numbers

You probably consider it a familiar fact that a number is rational if and only if its decimal representation terminates or repeats. In this subsection, we aim to prove part of this statement. In particular, we will show that, if \( x \) has a decimal expansion that infinitely repeats a string of digits, then \( x \) is rational. We work up to this goal by first considering some concrete examples.

**Example 10.4.2.** Consider the real number \( x \) with decimal representation 
\[
0.\overline{17} = 0.171717\ldots
\]
We claim that \( x \) is rational.

By definition,
\[
x = \frac{1}{10} + \frac{7}{10^2} + \frac{1}{10^3} + \frac{7}{10^4} + \ldots.
\]
We would like to group the terms in pairs and write
\[
x = \frac{17}{10^2} + \frac{17}{10^4} + \frac{17}{10^6} + \ldots = \sum_{n=1}^{\infty} 17 \left( \frac{1}{10^2} \right)^n. \tag{10.4.2}
\]
Is this grouping legitimate? It turns out that for arbitrary series, grouping the terms may change the convergence properties of the series or the value to which it converges. For series of positive terms, however, such grouping is legitimate. See the exercises following this example and see also the discussion of groupings and rearrangements in Chapter REF for a more extensive discussion of these issues.

Taking equation (10.4.2) as valid, we see that \( x \) equals a convergent geometric series with \( r = \frac{1}{100} \) and \( a = \frac{17}{100} \), and so
\[
x = \frac{\frac{17}{100}}{1 - \frac{1}{100}} = \frac{17}{99}.
\]

**Exercise 10.4.1.** The purpose of this exercise is to illustrate that, if the terms of a series are not all positive, grouping of terms may not be legitimate. Let \( a_n = (-1)^n \). Determine whether each of the following series converges or diverges.

(a) \( \sum_{n=0}^{\infty} a_n \)

(b) \( \sum_{n=0}^{\infty} (a_{2n} + a_{2n+1}) = (a_0 + a_1) + (a_2 + a_3) + (a_4 + a_5) + \ldots \)

(c) \( a_0 + \sum_{n=1}^{\infty} (a_{2n+1} + a_{2n}) = a_0 + (a_1 + a_2) + (a_3 + a_4) + (a_5 + a_6) + \ldots \)

**Exercise 10.4.2.** Let \( \{a_n\} \) be a sequence of non-negative real numbers. Consider \( b_n = a_{2n} + a_{2n+1} \). Prove that \( \sum a_n \) converges if and only if \( \sum b_n \) converges.

**Exercise 10.4.3.** In each case, find the rational number with the given decimal representation.
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(a) 0.123
(b) 0.1239

With these examples and exercises behind us, we prove a general result.

Proposition 10.4.2. Suppose \( x \) is a real number with decimal expansion

\[ 0.a_1 \ldots a_N\overline{a_{N+1} \ldots a_{N+p}}. \]

Then \( x \) is rational.

Proof. The proof is a uses the same idea as the example above; we express \( x \) as an infinite series and then observe that the series involves a convergent geometric series. Observe,

\[ x = \frac{a_1 \ldots a_N}{10^N} + \frac{a_{N+1} \ldots a_{N+p}}{10^{N+p}} + \frac{a_{N+1} \ldots a_{N+p}}{10^{N+2p}} + \ldots \]

\[ = \frac{a_1 \ldots a_N}{10^N} + \sum_{n=1}^{\infty} \frac{a_{N+1} \ldots a_{N+p}}{10^{N+np}}. \]

The last infinite series is a geometric series with

\[ a = \frac{a_{N+1} \ldots a_{N+p}}{10^{N+p}} \]

and \( r = 10^{-p} \). It thus converges to \( a/(1 - r) \) which is rational because \( a \) and \( r \) are rational. Because \( x \) is the sum of this number and the rational number \( \frac{a_1 \ldots a_N}{10^N} \), \( x \) is itself rational.

\[ \square \]

10.5 Problems

1. Determine whether each of the following series is convergent or divergent. You need not find the value of a convergent series.

(a) \( \sum_{n=0}^{\infty} \frac{2^n}{n!} \)
(b) \( \sum_{n=1}^{\infty} \frac{n}{(n+1)^2} \)
(c) \( \sum_{n=0}^{\infty} \frac{n!}{n^n} \)
(d) \( \sum_{n=0}^{\infty} \frac{n^n}{n!} \)
(e) \( \sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2 + 4} \)
2. For which real numbers $x$ does \( \sum_{n=0}^{\infty} \frac{(x - 2)^n}{2^n n^3} \) converge?

3. Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of real numbers.
   (a) Suppose \( \sum (a_n + b_n) \) converges. Must \( \sum a_n \) and \( \sum b_n \) converge?
   (b) Suppose \( \sum a_n b_n \) converges. Must \( \sum a_n \) and \( \sum b_n \) converge?

4. The Cantor middle-thirds set is constructed as follows: Let \( C_0 = [0,1] \). Then, for \( n > 0 \), obtain the set \( C_n \) from \( C_{n-1} \) by deleting from each closed interval making up \( C_{n-1} \) an open interval that is its “middle third.” Thus to create \( C_1 \), we remove \( \left( \frac{1}{3}, \frac{2}{3} \right) \) from \([0,1]\) and so \( C_1 = [0, \frac{1}{3}] \cup \left[ \frac{2}{3}, 1 \right] \). The Cantor set is then \( \cap C_n \).
   (a) Find \( C_2, C_3, C_4 \).
   (b) Let \( L_n \) be the length of the intervals removed to obtain \( C_n \) from \( C_{n-1} \). (Thus \( L_0 \) is not defined, and \( L_1 = \frac{1}{3} \), etc.) Find a simple formula for \( L_n \).
   (c) Find \( \sum_{n=1}^{\infty} L_n \). This number represents the sum of the lengths of all intervals removed to form the Cantor set.

5. Prove the Limit Comparison Test: Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of positive real numbers. Suppose \( \lim_{n \to \infty} a_n/b_n = L \), where \( L \) is a nonzero real number. Then \( \sum a_n \) converges if and only if \( \sum b_n \) converges.

6. Let \( p \) and \( q \) be polynomials and suppose that neither is the zero polynomial. Consider \( \sum_{n=n_0}^{\infty} \frac{p(n)}{q(n)} \) where \( n_0 \) is large enough that \( q(n) \neq 0 \) for all \( n \geq n_0 \). Such an \( n_0 \) exists because \( q \) has only finitely many zeros and we can take \( n_0 \) to be a natural number greater than the maximum of the zeros of \( q \). State and prove a theorem that relates the convergence or divergence of the series \( \sum_{n=n_0}^{\infty} \frac{p(n)}{q(n)} \) to the degrees of \( p \) and \( q \). (Remark: Although many approaches are possible, you may find the Limit Comparison Test from the previous exercise particularly helpful.)

7. You may have been taught in elementary or middle school to use the following procedure to convert a repeating decimal to a rational number. Suppose \( x = 0.a_1\ldots a_N.\bar{a_1}\ldots\bar{a_N} \). Then \( 10^N x = a_1\ldots a_N.\bar{a_1}\ldots\bar{a_N} \). Subtract the former from the latter, and solve for \( x \).
   (a) Use this procedure to write \( 0.\overline{4} \) and \( 0.\overline{123} \) as ratios of integers.
   (b) We claim the above procedure relies on a number of series about infinite series. Write a more detailed description of the procedure that justifies each step by stating explicitly what results about series are used.