# POLYNOMIAL IDENTITY RINGS AS RINGS OF FUNCTIONS

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ABSTRACT. We generalize the usual relationship between irreducible Zariski closed subsets of the affine space, their defining ideals, coordinate rings, and function fields, to a non-commutative setting, where "varieties" carry a $\mathrm{PGL}_n$-action, regular and rational "functions" on them are matrix-valued, "coordinate rings" are prime polynomial identity algebras, and "function fields" are central simple algebras of degree $n$. In particular, a prime polynomial identity algebra of degree $n$ is finitely generated if and only if it arises as the "coordinate ring" of a "variety" in this setting. For $n = 1$ our definitions and results reduce to those of classical affine algebraic geometry.

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## 1. Introduction

Polynomial identity rings (or PI-rings, for short) are often viewed as being "close to commutative"; they have large centers, and their structure (and in particular, their maximal spectra) have been successfully studied by geometric means (see the references at the end of the section). In this paper we revisit this subject from the point of view of classical affine algebraic geometry. We will show that the usual relationship between irreducible Zariski closed subsets of the affine space, their defining ideals, coordinate rings, and function fields, can be extended to the setting of PI-rings.

Before proceeding with the statements of our main results, we will briefly introduce the objects that will play the roles of varieties, defining ideals, coordinate rings, etc. Throughout this paper we will work over an algebraically closed base field $k$ of characteristic zero. We also fix an integer $n \geq 1$, which will be the PI-degree of most of the rings we will consider. We will write $\mathrm{M}_n$ for the matrix algebra $\mathrm{M}_n(k)$. The vector space of $m$-tuples of $n \times n$-matrices will be denoted by $(\mathrm{M}_n)^m$; we will always assume that $m \geq 2$. The group $\mathrm{PGL}_n$ acts on $(\mathrm{M}_n)^m$ by simultaneous conjugation. The $\mathrm{PGL}_n$-invariant dense open subset

$$U_{m,n} = \{(a_1, \dots, a_m) \in (\mathrm{M}_n)^m \mid a_1, \dots, a_m \text{ generate } \mathrm{M}_n \text{ as } k\text{-algebra}\}$$

of $(\mathrm{M}_n)^m$ will play the role of the affine space $\mathbb{A}^m$ in the sequel. (Note that $U_{m,1} = \mathbb{A}^m$.) The role of affine algebraic varieties will be played by $\mathrm{PGL}_n$-invariant closed subvarieties of $U_{m,n}$; for lack of a better term, we shall call such objects *$n$-varieties*; see Section 3. (Note that, in general, $n$-varieties are not affine in the usual sense.) The role of the polynomial ring $k[x_1, \dots, x_m]$ will be played by the algebra $G_{m,n} = k\{X_1, \dots, X_m\}$ of $m$ generic $n \times n$ matrices, see 2.4. Elements of $G_{m,n}$ may be thought of as $\mathrm{PGL}_n$-equivariant maps $(\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$; if $n = 1$ these are simply the polynomial maps $k^m \longrightarrow k$. Using these maps, we define, in a manner analogous to the commutative case, the associated ideal $\mathcal{I}(X) \subset G_{m,n}$, the PI-coordinate ring $k_n[X] = G_{m,n}/\mathcal{I}(X)$, and the central simple algebra $k_n(X)$ of rational functions on an irreducible $n$-variety $X \subset U_{m,n}$; see Definitions 3.1 and 7.1. We show that Hilbert's Nullstellensatz continues to hold in this context; see Section 5. We also define the notions of a regular map (and, in particular, an isomorphism) $X \longrightarrow Y$ and a rational map (and, in particular, a birational isomorphism) $X \dashrightarrow Y$ between $n$-varieties $X \subset U_{m,n}$ and $Y \subset U_{l,n}$; see Definitions 6.1 and 7.5.

In categorical language our main results can be summarized as follows. Let

*$Var_n$* be the category of irreducible $n$-varieties, with regular maps of $n$-varieties as morphisms (see Definition 6.1), and

*$PI_n$* be the category of finitely generated prime $k$-algebras of PI-degree $n$ (here the morphisms are the usual $k$-algebra homomorphisms).

1.1. **Theorem.** *The functor defined by*

$$\begin{array}{rcl} X & \mapsto & k_n[X] \\ (f\colon X \longrightarrow Y) & \mapsto & (f^*\colon k_n[Y] \longrightarrow k_n[X]) \end{array}$$

*is a contravariant equivalence of categories between $Var_n$ and $PI_n$.*

In particular, every finitely generated prime PI-algebra is the coordinate ring of a uniquely determined $n$-variety; see Theorem 6.4. For a proof of Theorem 1.1, see Section 6.

Every $n$-variety is, by definition, an algebraic variety with a generically free $\mathrm{PGL}_n$-action. It turns out that, up to birational isomorphism, the converse holds as well; see Lemma 8.1. To summarize our results in the birational context, let

$Bir_n$ be the category of irreducible generically free $\mathrm{PGL}_n$-varieties, with dominant rational $\mathrm{PGL}_n$-equivariant maps as morphisms, and

$CS_n$ be the category of central simple algebras $A$ of degree $n$, such that the center of $A$ is a finitely generated field extension of $k$. Morphisms in $CS_n$ are $k$-algebra homomorphisms (these are necessarily injective).

1.2. **Theorem.** *The functor defined by*

$$\begin{array}{rcl} X & \mapsto & k_n(X) \\ (g\colon X \dashrightarrow Y) & \mapsto & (g^*\colon k_n(Y) \hookrightarrow k_n(X)) \end{array}$$

*is a contravariant equivalence of categories between $Bir_n$ and $CS_n$.*

Here for any $\mathrm{PGL}_n$-variety $X$, $k_n(X)$ denotes the $k$-algebra of $\mathrm{PGL}_n$-equivariant rational maps $X \dashrightarrow \mathrm{M}_n$ (with addition and multiplication induced from $\mathrm{M}_n$), see Definition 8.2. If $X$ is an irreducible $n$-variety then $k_n(X)$ is the total ring of fractions of $k_n[X]$, as in the commutative case; see Definition 7.1 and Proposition 7.3. Note also that $g^*(\alpha)$ stands for $\alpha \circ g$ (again, as in the commutative case). For a proof of Theorem 1.2, see Section 8.

Note that for $n = 1$, Theorems 1.1 and 1.2 are classical results of affine algebraic geometry; cf., e.g., [13, Corollary 3.8 and Theorem 4.4].

It is well known that central simple algebras $A/K$ of degree $n$ are in a natural bijection with $n-1$-dimensional Brauer-Severi varieties over $K$ and (if $K/k$ is a finitely generated field extension) with generically free $\mathrm{PGL}_n$-varieties $X/k$ such that $k(X)^{\mathrm{PGL}_n} \simeq K$. Indeed, all three are parametrized by the Galois cohomology set $H^1(K, \mathrm{PGL}_n)$; for details, see Section 9. Theorem 1.2 may thus be viewed as a way of explicitly identifying generically free $\mathrm{PGL}_n$-varieties $X$ with central simple algebras $A$, without going through $H^1(K, \mathrm{PGL}_n)$. The fact that the map $X \mapsto k_n(X)$ is bijective was proved in [24, Proposition 8.6 and Lemma 9.1]; here we give a more conceptual proof and show that this map is, in fact, a contravariant functor. In Section 9 we show how to construct the Brauer-Severi variety of $A$ directly from $X$.

Many of the main themes of this paper (such as the use of $\mathrm{PGL}_n$-actions, generic matrices, trace rings, and affine geometry in the study of polynomial identity algebras) were first systematically explored in the pioneering work of Amitsur, Artin and Procesi [2, 4, 19, 20, 21] in the 1960s and 70s. Our approach here was influenced by these, as well as other papers in this area, such as [3, 5, 7, 23]. In particular, Proposition 5.3 is similar in spirit to Amitsur's Nullstellensatz [1] (cf. Remark 5.6), and Theorem 1.1 to Procesi's functorial description of algebras satisfying the $n$th Cayley-Hamilton identity [22, Theorem 2.6]. (For a more geometric statement of Procesi's theorem, along the lines of our Theorem 1.1, see [15, Theorem 2.3].) We thank L. Small and the referee for bringing some of these connections to our attention.

## 2. Preliminaries

In this section we review some known results about matrix invariants and related PI-theory.

**2.1. Matrix invariants.** Consider the diagonal action of $\mathrm{PGL}_n$ on the space $(\mathrm{M}_n)^m$ of $m$-tuples of $n \times n$-matrices. We shall denote the ring of invariants for this action by $C_{m,n} = k[(\mathrm{M}_n)^m]^{\mathrm{PGL}_n}$, and the affine variety $\mathrm{Spec}(C_{m,n})$ by $Q_{m,n}$. It is known that $C_{m,n}$ is generated as a $k$-algebra, by elements of the form $(A_1, \dots, A_m) \mapsto \mathrm{tr}(M)$, where $M$ is a monomial in $A_1, \dots, A_m$ (see [21]); however, we shall not need this fact in the sequel. The inclusion $C_{m,n} \hookrightarrow k[(\mathrm{M}_n)^m]$ of $k$-algebras induces the categorical quotient map

$$\pi\colon (\mathrm{M}_n)^m \longrightarrow Q_{m,n}\,. \tag{2.2}$$

We shall need the following facts about this map in the sequel. Recall from the introduction that we always assume $m \geq 2$, the base field $k$ is algebraically closed and of characteristic zero, and

$$U_{m,n} = \{(a_1, \dots, a_m) \in (\mathrm{M}_n)^m \mid a_1, \dots, a_m \text{ generate } \mathrm{M}_n \text{ as } k\text{-algebra }\}.$$

2.3. **Proposition.** (a) *If $x \in U_{m,n}$ then $\pi^{-1}(\pi(x))$ is the $\mathrm{PGL}_n$-orbit of $x$.*

(b) *$\mathrm{PGL}_n$-orbits in $U_{m,n}$ are closed in $(\mathrm{M}_n)^m$.*

(c) *$\pi$ maps closed $\mathrm{PGL}_n$-invariant sets in $(\mathrm{M}_n)^m$ to closed sets in $Q_{m,n}$.*

(d) *$\pi(U_{m,n})$ is Zariski open in $Q_{m,n}$.*

(e) *If $Y$ is a closed irreducible subvariety of $Q_{m,n}$ then $\pi^{-1}(Y) \cap U_{m,n}$ is irreducible in $(\mathrm{M}_n)^m$.*

*Proof.* (a) is proved in [4, (12.6)].

(b) is an immediate consequence of (a).

(c) is a special case of [18, Corollary to Theorem 4.6].

(d) It is easy to see that $U_{m,n}$ is Zariski open in $(\mathrm{M}_n)^m$. Let $U_{m,n}^c$ be its complement in $(\mathrm{M}_n)^m$. By (c), $\pi(U_{n,m}^c)$ is closed in $Q_{m,n}$ and by (a), $\pi(U_{m,n}) = Q_{m,n} \setminus \pi(U_{m,n}^c)$.

(e) Let $V_1, \ldots, V_r$ be the irreducible components of $\pi^{-1}(Y)$ in $(\mathrm{M}_n)^m$. Since $\mathrm{PGL}_n$ is connected, each $V_i$ is $\mathrm{PGL}_n$-invariant. By part (c), $\pi(V_1)$, $\ldots$, $\pi(V_r)$ are closed subvarieties of $Q_{m,n}$ covering $Y$. Since $Y$ is irreducible, we may assume, after possibly renumbering $V_1, \ldots, V_r$, that $Y = \pi(V_1)$. It suffices to show that

$$\pi^{-1}(Y) \cap U_{m,n} = V_1 \cap U_{m,n} \,;$$

since $V_1 \cap U_{m,n}$ is irreducible, this will complete the proof of part (e). As $V_1$ is an irreducible component of $\pi^{-1}(Y)$, we clearly have

$$V_1 \cap U_{m,n} \subset \pi^{-1}(Y) \cap U_{m,n} \,.$$

To prove the opposite inclusion, let $y \in \pi^{-1}(Y) \cap U_{m,n}$. We want to show $y \in V_1$. Since $\pi(y) \in Y = \pi(V_1)$, there is a point $v \in V_1$ such that $\pi(y) = \pi(v)$. That is, $v$ lies in $\pi^{-1}(\pi(y))$, which, by part (a), is the $\mathrm{PGL}_n$-orbit of $y$. In other words, $y = g \cdot v$ for some $g \in \mathrm{PGL}_n$. Since $V_1$ is $\mathrm{PGL}_n$-invariant, this shows that $y \in V_1$, as claimed. □

**2.4. The ring of generic matrices and its trace ring.** Consider $m$ generic matrices

$$X_1 = (x_{ij}^{(1)})_{i,j=1,\ldots,n} \,, \ldots, \, X_m = (x_{ij}^{(m)})_{i,j=1,\ldots,n} \,,$$

where $x_{ij}^{(h)}$ are $mn^2$ independent variables over the base field $k$. The $k$-subalgebra generated by $X_1, \ldots, X_m$ inside $\mathrm{M}_n(k[x_{ij}^{(h)}])$ is called *the algebra of $m$ generic $n \times n$-matrices* and is denoted by $G_{m,n}$. If the values of $n$ and $m$ are clear from the context, we will simply refer to $G_{m,n}$ as the algebra of generic matrices.

The trace ring of $G_{m,n}$ is denoted by $T_{m,n}$; it is the $k$-algebra generated, inside $\mathrm{M}_n(k[x_{ij}^{(h)}])$ by elements of $G_{m,n}$ and their traces. Elements of $\mathrm{M}_n(k[x_{ij}^{(h)}])$ can be naturally viewed as regular (i.e., polynomial) maps $(\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$. (Note that $k[x_{ij}^{(h)}]$ in the coordinate ring of $(\mathrm{M}_n)^m$.) Here $\mathrm{PGL}_n$ acts on both $(\mathrm{M}_n)^m$ and $\mathrm{M}_n$ by simultaneous conjugation; Procesi [21, Section 1.2] noticed that $T_{m,n}$ consists precisely of those maps $(\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$ that are equivariant with respect to this action. (In particular, the $i$-th generic matrix $X_i$ is the projection to the $i$-th component.) In this way the invariant ring $C_{m,n} = k[(\mathrm{M}_n)^m]^{\mathrm{PGL}_n}$ which we considered in 2.1, is naturally identified with the center of $T_{m,n}$ via $f \mapsto f I_{n \times n}$.

We now recall the following definitions.

2.5. **Definition.** (a) A prime PI-ring is said to have PI-degree $n$ if its total ring of fractions is a central simple algebra of degree $n$.

(b) Given a ring $R$, $\mathrm{Spec}_n(R)$ is defined as the set of prime ideals $J$ of $R$ such that $R/J$ has PI-degree $n$; cf. e.g., [19, p. 58] or [26, p. 75].

The following lemma shows that $\mathrm{Spec}_n(G_{m,n})$ and $\mathrm{Spec}_n(T_{m,n})$ are closely related.

2.6. **Lemma.** *The assignment* $J \mapsto J \cap G_{m,n}$ *defines a bijective correspondence between* $\mathrm{Spec}_n(T_{m,n})$ *and* $\mathrm{Spec}_n(G_{m,n})$*. In addition, for any prime ideal* $J \in \mathrm{Spec}_n(T_{m,n})$*, we have the following:*

(a) *The natural projection* $\phi \colon T_{m,n} \longrightarrow T_{m,n}/J$ *is trace-preserving, and* $T_{m,n}/J$ *is the trace ring of* $G_{m,n}/(J \cap G_{m,n})$*.*
(b) $\mathrm{tr}(p) \in J$ *for every* $p \in J$*.*

*Proof.* The first assertion and part (a) are special cases of results proved in [3, §2]. Part (b) follows from (a), since for any $p \in J$, $\phi(\mathrm{tr}(p)) = \mathrm{tr}(\phi(p)) = \mathrm{tr}(0) = 0$. In other words, $\mathrm{tr}(p) \in \mathrm{Ker}(\phi) = J$, as claimed. □

**2.7. Central polynomials.** We need to construct central polynomials with certain non-vanishing properties. We begin by recalling two well-known facts from the theory of rings satisfying polynomial identities.

2.8. **Proposition.** (a) *Let* $k\{x_1, \dots, x_m\}$ *be the free associative algebra. Consider the natural homomorphism* $k\{x_1, \dots, x_m\} \longrightarrow G_{m,n}$*, taking* $x_i$ *to the* $i$*-th generic matrix* $X_i$*. The kernel of this homomorphism is precisely the ideal of polynomial identities of* $n \times n$*-matrices in* $m$ *variables.*

(b) *Since* $k$ *is an infinite field, all prime* $k$*-algebras of the same PI-degree satisfy the same polynomial identities* (*with coefficients in* $k$)*.*

*Proof.* See [19, pp. 20-21] or [26, p. 16] for a proof of part (a) and [27, pp. 106-107] for a proof of part (b). □

For the convenience of the reader and lack of a suitable reference, we include the following definition.

2.9. **Definition.** An ($m$-variable) *central polynomial for* $n \times n$ *matrices* is an element $p = p(x_1, \dots, x_m) \in k\{x_1, \dots, x_m\}$ satisfying one of the following equivalent conditions:

(a) $p$ is a polynomial identity of $\mathrm{M}_{n-1}$, and the evaluations of $p$ in $\mathrm{M}_n$ are central (i.e., scalar matrices) but not identically zero.
(b) $p$ is a polynomial identity for all prime $k$-algebras of PI-degree $n-1$, and the evaluations of $p$ in every prime $k$-algebra of PI-degree $n$ are central but not identically zero.
(c) $p$ is a polynomial identity for all prime $k$-algebras of PI-degree $n-1$, and the canonical image of $p$ in $G_{m,n}$ is a nonzero central element.
(d) The constant coefficient of $p$ is zero, and the canonical image of $p$ in $G_{m,n}$ is a nonzero central element.

That the evaluations of $p$ in an algebra $A$ are central is equivalent to saying that $x_{m+1}p - px_{m+1}$ is a polynomial identity for $A$, where $x_{m+1}$ is another free variable. Thus the equivalence of (a)—(c) easily follows from Proposition 2.8. The equivalence of (c) and (d) follows from [19, p. 172]. The existence of central polynomials for $n \times n$-matrices was established independently by Formanek and Razmyslov; see [11]. Because of Proposition 2.8(a), one can think of $m$-variable central polynomials of $n\times n$ matrices as nonzero central elements of $G_{m,n}$ (with zero constant coefficient).

The following lemma, establishing the existence of central polynomials with certain non-vanishing properties, will be repeatedly used in the sequel.

2.10. **Lemma.** *Let $A_1,\dots,A_r \in U_{m,n}$. Then there exists a central polynomial $s = s(X_1,\dots,X_m) \in G_{m,n}$ for $n\times n$-matrices such that $s(A_i)\neq 0$ for $i = 1,\dots,r$. In other words, each $s(A_i)$ is a non-zero scalar matrix in $\mathrm{M}_n$.*

*Proof.* First note that if $A_i$ and $A_j$ are in the same $\mathrm{PGL}_n$-orbit then $s(A_i) = s(A_j)$. Hence, we may remove $A_j$ from the set $\{A_1,\dots,A_r\}$. After repeating this process finitely many times, we may assume that no two of the points $A_1,\dots,A_r$ lie in the same $\mathrm{PGL}_n$-orbit.

By the above-mentioned theorem of Formanek and Razmyslov, there exists a central polynomial $c = c(X_1,\dots,X_N) \in G_{N,n}$ for $n\times n$-matrices. Choose $b_1,\dots,b_N \in \mathrm{M}_n$ such that $c(b_1,\dots,b_N)\neq 0$. We now define $s$ by modifying $c$ as follows:

$$s(X_1,\dots,X_m) = c\big(p_1(X_1,\dots,X_m),\dots,p_N(X_1,\dots,X_m)\big)\,,$$

where the elements $p_j = p_j(X_1,\dots,X_m) \in G_{m,n}$ will be chosen below so that for every $j = 1,\dots,N$,

$$p_j(A_1) = p_j(A_2) = \dots = p_j(A_r) = b_j \in \mathrm{M}_n\,. \tag{2.11}$$

We first check that this polynomial has the desired properties. Being an evaluation of a central polynomial for $n\times n$ matrices, $s$ is a central element in $G_{m,n}$ and a polynomial identity for all prime $k$-algebras of PI-degree $n-1$. Moreover, $s(A_i) = c(b_1,\dots,b_N)\neq 0$ for every $i = 1,\dots,r$. Thus $s$ itself is a central polynomial for $n\times n$ matrices. Consequently, $s(A_i)$ is a central element in $\mathrm{M}_n$, i.e., a scalar matrix.

It remains to show that $p_1,\dots,p_N \in G_{m,n}$ can be chosen so that (2.11) holds. Consider the representation $\phi_i\colon G_{m,n}\longrightarrow \mathrm{M}_n$ given by $p\longmapsto p(A_i)$. Since each $A_i$ lies in $U_{m,n}$, each $\phi_i$ is surjective. Moreover, by our assumption on $A_1,\dots,A_r$, no two of them are conjugate under $\mathrm{PGL}_n$, i.e., no two of the representations $\phi_i$ are equivalent. The kernels of the $\phi_i$ are thus pairwise distinct by [4, Theorem (9.2)]. Hence the Chinese Remainder Theorem tells us that $\phi_1\oplus\dots\oplus\phi_r\colon G_{m,n}\longrightarrow(\mathrm{M}_n)^r$ is surjective; $p_j$ can now be chosen to be any preimage of $(b_j,\dots,b_j)\in(\mathrm{M}_n)^r$. This completes the proof of Lemma 2.10. □

## 3. Definition and first properties of $n$-varieties

3.1. **Definition.** (a) An *$n$-variety* $X$ is a closed $\mathrm{PGL}_n$-invariant subvariety of $U_{m,n}$ for some $m\geq 2$. In other words, $X = \overline{X}\cap U_{m,n}$, where $\overline{X}$ is the Zariski closure of $X$ in $(\mathrm{M}_n)^m$. Note that $X$ is a generically free $\mathrm{PGL}_n$-variety (in fact, for every $x\in X$, the stabilizer of $x$ in $\mathrm{PGL}_n$ is trivial).

(b) Given a subset $S\subset G_{m,n}$ (or $S\subset T_{m,n}$), we define its zero locus as

$$\mathcal{Z}(S) = \{a = (a_1,\dots,a_m)\in U_{m,n}\,|\,p(a) = 0,\ \ \forall p\in S\}\,.$$

Of course, $\mathcal{Z}(S) = \mathcal{Z}(J)$, where $J$ is the 2-sided ideal of $G_{m,n}$ (or $T_{m,n}$) generated by $S$. Conversely, given an $n$-variety $X \subset U_{m,n}$ we define its ideal as

$$\mathcal{I}(X) = \{p \in G_{m,n} \mid p(a) = 0, \ \ \forall a \in X\}\,.$$

Similarly we define the ideal of $X$ in $T_{m,n}$, as

$$\mathcal{I}_T(X) = \{p \in T_{m,n} \mid p(a) = 0, \ \ \forall a \in X\}\,.$$

Note that $\mathcal{I}(X) = \mathcal{I}_T(X) \cap G_{m,n}$.

(c) The *polynomial identity coordinate ring* (or *PI-coordinate ring*) of an $n$-variety $X$ is defined as $G_{m,n}/\mathcal{I}(X)$. We denote this ring by $k_n[X]$.

3.2. **Remark.** Elements of $k_n[X]$ may be viewed as $\mathrm{PGL}_n$-equivariant morphisms $X \longrightarrow \mathrm{M}_n$. The example below shows that not every $\mathrm{PGL}_n$-equivariant morphism $X \longrightarrow \mathrm{M}_n(k)$ is of this form. On the other hand, if $X$ is irreducible, we will later prove that every $\mathrm{PGL}_n$-equivariant *rational* map $X \dashrightarrow \mathrm{M}_n(k)$ lies in the total ring of fractions of $k_n[X]$; see Proposition 7.3.

3.3. **Example.** Recall that $U_{2,2}$ is the open subset of $\mathrm{M}_{2,2}$ defined by the inequality $c(X_1, X_2) \neq 0$, where

$$c(X_1,X_2) = \big(2\operatorname{tr}(X_1^2) - \operatorname{tr}(X_1)^2\big)\big(2\operatorname{tr}(X_2^2) - \operatorname{tr}(X_2)^2\big) - \big(2\operatorname{tr}(X_1X_2) - \operatorname{tr}(X_1)\det(X_2)\big)^2\,;$$

see, e.g., [12, p. 191]. Thus for $X = U_{2,2}$, the $\mathrm{PGL}_n$-equivariant morphism $f\colon X \longrightarrow \mathrm{M}_2$ given by $(X_1, X_2) \longmapsto \frac{1}{c(X_1,X_2)} I_{2\times 2}$ is not in $k_2[X] = G_{2,2}$ (and not even in $T_{2,2}$).

3.4. **Remark.** (a) Let $J$ be an ideal of $G_{m,n}$. Then the points of $\mathcal{Z}(J)$ are in bijective correspondence with the surjective $k$-algebra homomorphisms $\phi\colon G_{m,n} \longrightarrow \mathrm{M}_n$ such that $J \subset \operatorname{Ker}(\phi)$ (or equivalently, with the surjective $k$-algebra homomorphisms $G_{m,n}/J \longrightarrow \mathrm{M}_n$). Indeed, given $a \in \mathcal{Z}(J)$, we associate to it the homomorphism $\phi_a$ given by $\phi_a\colon p \mapsto p(a)$. Conversely, a surjective homomorphism $\phi\colon G_{m,n} \longrightarrow \mathrm{M}_n$ such that $J \subset \operatorname{Ker}(\phi)$ gives rise to the point

$$a_\phi = (\phi(X_1), \dots, \phi(X_m)) \in \mathcal{Z}(J)\,,$$

where $X_i \in G_{m,n}$ is the $i$-th generic matrix in $G_{m,n}$. One easily checks that the assignments $a \mapsto \phi_a$ and $\phi \mapsto a_\phi$ are inverse to each other.

(b) The claim in part (a) is also true for an ideal $J$ of $T_{m,n}$. That is, the points of $\mathcal{Z}(J)$ are in bijective correspondence with the surjective $k$-algebra homomorphisms $\phi\colon T_{m,n} \longrightarrow \mathrm{M}_n$ such that $J \subset \operatorname{Ker}(\phi)$ (or equivalently, with the surjective $k$-algebra homomorphisms $T_{m,n}/J \longrightarrow \mathrm{M}_n$). The proof goes through without changes.

3.5. **Lemma.** *Let $a = (a_1, \dots, a_m) \in U_{m,n}$ and let $J$ be an ideal of $G_{m,n}$ (or of $T_{mn}$). Let*

$$J(a) = \{j(a) \mid j \in J\} \subset \mathrm{M}_n\,.$$

*Then either* $J(a) = (0)$ *(i.e,* $a \in \mathcal{Z}(J)$*) or* $J(a) = \mathrm{M}_n$*.*

*Proof.* Since $a_1, \dots, a_m$ generate $\mathrm{M}_n$, $\phi_a(J)$ is a (2-sided) ideal of $\mathrm{M}_n$. Since $\mathrm{M}_n$ is simple, the lemma follows. □

3.6. **Lemma.** (a) $\mathcal{Z}(J) = \mathcal{Z}(J \cap G_{m,n})$ *for every ideal* $J \subset T_{m,n}$*.*

(b) *If* $X \subset U_{m,n}$ *is an* $n$*-variety, then* $X = \mathcal{Z}(\mathcal{I}(X)) = \mathcal{Z}(\mathcal{I}_T(X))$*.*

*Proof.* (a) Clearly, $\mathcal{Z}(J) \subset \mathcal{Z}(J \cap G_{m,n})$. To prove the opposite inclusion, assume the contrary: there exists a $y \in U_{m,n}$ such that $p(y) = 0$ for every $p \in J \cap G_{m,n}$ but $f(y) \neq 0$ for some $f \in J$. By Lemma 2.10 there exists a central polynomial $s \in G_{m,n}$ for $n \times n$-matrices such that $s(y) \neq 0$. By [28, Theorem 1], $p = s^i f$ lies in $G_{m,n}$ (and hence, in $J \cap G_{m,n}$) for some $i \geq 0$. Our choice of $y$ now implies $0 = p(y) = s^i(y) f(y)$. Since $s(y)$ is a non-zero element of $k$, we conclude that $f(y) = 0$, a contradiction.

(b) Clearly $X \subset \mathcal{Z}(\mathcal{I}_T(X)) \subset \mathcal{Z}(\mathcal{I}(X))$. Part (a) (with $J = \mathcal{I}_T(X)$) tells us that $\mathcal{Z}(\mathcal{I}(X)) = \mathcal{Z}(\mathcal{I}_T(X))$. It thus remains to be shown that $\mathcal{Z}(\mathcal{I}_T(X)) \subset X$. Assume the contrary: there exists a $z \in U_{m,n}$ such that $p(z) = 0$ for every $p \in \mathcal{I}_T(X)$ but $z \notin X$. Since $X = \overline{X} \cap U_{m,n}$, where $\overline{X}$ is the closure of $X$ in $(\mathrm{M}_n)^m$, we conclude that $z \notin \overline{X}$. Let $C = \mathrm{PGL}_n \cdot z$ be the orbit of $z$ in $(\mathrm{M}_n)^m$. Since $z \in U_{m,n}$, $C$ is closed in $(\mathrm{M}_n)^m$; see Proposition 2.3(b). Thus $C$ and $\overline{X}$ are disjoint closed $\mathrm{PGL}_n$-invariant subsets of $(\mathrm{M}_n)^m$. By [16, Corollary 1.2], there exists a $\mathrm{PGL}_n$-invariant regular function $f \colon (\mathrm{M}_n)^m \longrightarrow k$ such that $f \equiv 0$ on $\overline{X}$ but $f \not\equiv 0$ on $C$. The latter condition is equivalent to $f(z) \neq 0$. Identifying elements of $k$ with scalar matrices in $\mathrm{M}_n$, we may view $f$ as a central element of $T_{m,n}$. So $f \in \mathcal{I}_T(X)$ but $f(z) \neq 0$, contradicting our assumption. □

## 4. Irreducible $n$-varieties

Of particular interest to us will be *irreducible* $n$-varieties. Here "irreducible" is understood with respect to the $n$-Zariski topology on $U_{m,n}$, where the closed subsets are the $n$-varieties. However, since $\mathrm{PGL}_n$ is a connected group, each irreducible component of $X$ in the usual Zariski topology is $\mathrm{PGL}_n$-invariant. Consequently, $X$ is irreducible in the $n$-Zariski topology if and only if it is irreducible in the usual Zariski topology.

4.1. **Lemma.** *Let* $\emptyset \neq X \subset U_{m,n}$ *be an* $n$*-variety. The following are equivalent:*

(a) $X$ *is irreducible.*
(b) $\mathcal{I}_T(X)$ *is a prime ideal of* $T_{m,n}$*.*
(c) $\mathcal{I}(X)$ *is a prime ideal of* $G_{m,n}$*.*
(d) $k_n[X]$ *is a prime ring.*

*Proof.* (a) ⇒ (b): Suppose that $\mathcal{I}_T(X)$ is not prime, i.e., there are ideals $J_1$ and $J_2$ such that $J_1 \cdot J_2 \subset \mathcal{I}_T(X)$ but $J_1, J_2 \not\subset \mathcal{I}_T(X)$. We claim that $X$ is not irreducible. Indeed, by Lemma 3.5, $X \subset \mathcal{Z}(J_1) \cup \mathcal{Z}(J_2)$. In other words, we can write $X = X_1 \cup X_2$, as a union of two $n$-varieties, where

$X_1 = \mathcal{Z}(J_1) \cap X$ and $X_2 = \mathcal{Z}(J_2) \cap X$. It remains to be shown that $X_i \neq X$ for $i = 1, 2$. Indeed, if say, $X_1 = X$ then every element of $J_1$ vanishes on all of $X$, so that $J_1 \subset \mathcal{I}_T(X)$, contradicting our assumption.

(b) $\Rightarrow$ (c): Clear, since $\mathcal{I}(X) = \mathcal{I}_T(X) \cap G_{m,n}$.

(c) $\Leftrightarrow$ (d): $\mathcal{I}(X) \subset G_{m,n}$ is, by definition, a prime ideal if and only if $k_n[X] = G_{m,n}/\mathcal{I}(X)$ is a prime ring.

(c) $\Rightarrow$ (a): Assume $\mathcal{I}(X)$ is prime and $X = X_1 \cup X_2$ is a union of two $n$-varieties in $U_{m,n}$. Our goal is to show that $X = X_1$ or $X = X_2$. Indeed, $\mathcal{I}(X_1) \cdot \mathcal{I}(X_2) \subset \mathcal{I}(X)$ implies $\mathcal{I}(X_i) \subset \mathcal{I}(X)$ for $i = 1$ or 2. Taking the zero loci and using Lemma 3.6(b), we obtain

$$X_i = \mathcal{Z}(\mathcal{I}(X_i)) \supset \mathcal{Z}(\mathcal{I}(X)) = X \, ,$$

as desired. □

4.2. **Proposition.** *Let $J \in \operatorname{Spec}_n(T_{m,n})$. Then*

(a) $\mathcal{Z}(J) = \mathcal{Z}(J \cap C_{m,n})$

(b) $\mathcal{Z}(J)$ *is irreducible.*

*Proof.* (a) Clearly $\mathcal{Z}(J \cap C_{m,n}) \supset \mathcal{Z}(J)$. To prove the opposite inclusion, suppose $a \in \mathcal{Z}(J \cap C_{m,n})$ and consider the evaluation map $\phi_a \colon T_{m,n} \longrightarrow \mathrm{M}_n$ given by $\phi_a(p) = p(a)$.

Recall that $\phi_a$ is trace-preserving (see, e.g., [3, Theorem 2.2]). Since $\operatorname{tr}(j) \in J \cap C_{m,n}$ for every $j \in J$, we see that $\operatorname{tr}(j(a)) = 0$ for every $j \in J$. By Lemma 3.5 this implies that $a \in \mathcal{Z}(J)$, as claimed.

(b) Consider the categorical quotient map $\pi \colon (\mathrm{M}_n)^m \longrightarrow Q_{m,n}$ for the $\mathrm{PGL}_n$-action on $(\mathrm{M}_n)^m$. Recall that $C_{m,n} = k[Q_{m,n}]$ is the coordinate ring of $Q_{m,n}$. Note that elements of $C_{m,n}$ may be viewed in two ways: as regular functions on $Q_{m,n}$ or (after composing with $\pi$) as a $\mathrm{PGL}_n$-invariant regular function on $(\mathrm{M}_n)^m$. Let $Y \subset Q_{m,n}$ be the zero locus of $J \cap C_{m,n}$ in $Q_{m,n}$. Then by part (a),

$$\mathcal{Z}(J) = \mathcal{Z}(J \cap C_{m,n}) = \pi^{-1}(Y) \cap U_{m,n} \, .$$

Since $J$ is a prime ideal of $T_{m,n}$, $J \cap C_{m,n}$ is a prime ideal of $C_{m,n}$; see, e.g., [19, Theorem II.6.5(1)]. Hence, $Y$ is irreducible. Now by Proposition 2.3(e), we conclude that $\mathcal{Z}(J) = \pi^{-1}(Y) \cap U_{m,n}$ is also irreducible, as claimed. □

4.3. **Corollary.** *If $J_0 \in \operatorname{Spec}_n(G_{m,n})$, then $\mathcal{Z}(J_0)$ is irreducible.*

*Proof.* By Lemma 2.6, $J_0 = J \cap G_{m,n}$ for some $J \in \operatorname{Spec}_n(T_{m,n})$. By Lemma 3.6(a), $\mathcal{Z}(J_0) = \mathcal{Z}(J)$, and by Proposition 4.2(b), $\mathcal{Z}(J)$ is irreducible. □

## 5. The Nullstellensatz for prime ideals

5.1. **Proposition.** (Weak form of the Nullstellensatz) *Let $A$ denote the algebra $G_{m,n}$ or $T_{m,n}$, and let $J$ be a prime ideal of $A$. Then $\mathcal{Z}(J) \neq \emptyset$ if and only if $A/J$ has PI-degree $n$.*

Note that for $n = 1$, Proposition 5.1 reduces to the usual (commutative) weak Nullstellensatz (which is used in the proof of Proposition 5.1). Indeed, a prime ring of PI-degree 1 is simply a nonzero commutative domain; in this case $G_{m,1}/J = k[x_1, \ldots, x_m]/J$ has PI-degree 1 if and only if $J \neq k[x_1, \ldots, x_m]$.

*Proof.* First assume that $\mathcal{Z}(J) \neq \emptyset$. Since $A$ has PI-degree $n$, its quotient $A/J$ clearly has PI-degree $\leq n$. To show $\mathrm{PIdeg}(A/J) \geq n$, recall that a point $a = (a_1, \ldots, a_m) \in Z(J)$ gives rise to a surjective $k$-algebra homomorphism $A/J \longrightarrow \mathrm{M}_n$; see Remark 3.4.

Conversely, assume that $R = A/J$ is a $k$-algebra of PI-degree $n$. Note that $R$ is a Jacobson ring (i.e., the intersection of its maximal ideals is zero), and that it is a Hilbert $k$-algebra (i.e., every simple homomorphic image is finite-dimensional over $k$ and thus a matrix algebra over $k$), see [2, Corollary 1.2]. So if $c$ is a nonzero evaluation in $R$ of a central polynomial for $n \times n$-matrices, there is some maximal ideal $M$ of $R$ not containing $c$. Then $R/M \simeq \mathrm{M}_n$, and we are done in view of Remark 3.4. □

5.2. **Corollary.** *For any irreducible $n$-variety $X$, $T_{m,n}/\mathcal{I}_T(X)$ is the trace ring of the prime $k$-algebra $k_n[X]$.*

*Proof.* By Lemma 4.1 and Proposition 5.1, $\mathcal{I}_T(X)$ is a prime ideal of $T_{m,n}$ of PI-degree $n$. Consequently, $\mathcal{I}(X) = \mathcal{I}_T(X) \cap G_{m,n}$ is a prime ideal of $G_{m,n}$, and the desired conclusion follows from Lemma 2.6(a). □

5.3. **Proposition.** (Strong form of the Nullstellensatz)

(a) $\mathcal{I}(\mathcal{Z}(J_0)) = J_0$ *for every* $J_0 \in \mathrm{Spec}_n(G_{m,n})$.

(b) $\mathcal{I}_T(\mathcal{Z}(J)) = J$ *for every* $J \in \mathrm{Spec}_n(T_{m,n})$.

For $n = 1$ both parts reduce to the usual (commutative) strong form of the Nullstellensatz for prime ideals (which is used in the proof of Proposition 5.3).

*Proof.* We begin by reducing part (a) to part (b). Indeed, by Lemma 2.6, $J_0 = J \cap G_{m,n}$ for some $J \in \mathrm{Spec}_n(T_{m,n})$. Now

$$\mathcal{I}(\mathcal{Z}(J_0)) \overset{(1)}{=} \mathcal{I}(\mathcal{Z}(J)) = \mathcal{I}_T(\mathcal{Z}(J)) \cap G_{m,n} \overset{(2)}{=} J \cap G_{m,n} = J_0 \,,$$

where (1) follows from Lemma 3.6(a) and (2) follows from part (b).

It thus remains to prove (b). Let $X = \mathcal{Z}(J)$. Then $X \neq \emptyset$ (see Proposition 5.1), $X$ is irreducible (see Proposition 4.2(b)) and hence $\mathcal{I}_T(X)$ is a prime ideal of $T_{m,n}$ (see Lemma 4.1). Clearly $J \subset \mathcal{I}_T(X)$; our goal is to show that $J = \mathcal{I}_T(X)$. In fact, we only need to check that

$$J \cap C_{m,n} = \mathcal{I}_T(X) \cap C_{m,n} \,. \tag{5.4}$$

Indeed, suppose (5.4) is established. Choose $p \in \mathcal{I}_T(X)$; we want to show that $p \in J$. For every $q \in T_{m,n}$ we have $pq \in I_T(X)$ and thus

$$\mathrm{tr}(p \cdot q) \in \mathcal{I}_T(X) \cap C_{m,n} = J \cap C_{m,n} \,,$$

see Lemma 2.6(b). Hence, if we denote the images of $p$ and $q$ in $T_{m,n}/J$ by $\overline{p}$ and $\overline{q}$ respectively, Lemma 2.6(a) tells us that $\operatorname{tr}(\overline{p} \cdot \overline{q}) = 0$ in $T_{m,n}/J$ for every $\overline{q} \in T_{m,n}/J$. Consequently, $\overline{p} = 0$, i.e., $p \in J$, as desired.

We now turn to proving (5.4). Consider the categorical quotient map $\pi\colon (\mathrm{M}_n)^m \longrightarrow Q_{m,n}$ for the $\mathrm{PGL}_n$-action on $(\mathrm{M}_n)^m$; here $C_{m,n} = k[Q_{m,n}]$ is the coordinate ring of $Q_{m,n}$. Given an ideal $H \subset C_{m,n}$, denote its zero locus in $Q_{m,n}$ by

$$\mathcal{Z}_0(H) = \{a \in Q_{m,n} \mid h(a) = 0 \;\forall\, h \in H\}\,.$$

Since $\mathcal{Z}(J \cap C_{m,n}) = \mathcal{Z}(J) = X$ in $U_{m,n}$ (see Proposition 4.2(a)), we conclude that $\mathcal{Z}_0(J \cap C_{m,n}) \cap \pi(U_{m,n}) = \pi(X)$. On the other hand, since $\pi(U_{m,n})$ is Zariski open in $Q_{m,n}$ (see Lemma 2.3(d)), and $J \cap C_{m,n}$ is a prime ideal of $C_{m,n}$ (see [19, Theorem II.6.5(1)]), we have

$$\mathcal{Z}_0(J \cap C_{m,n}) = \overline{\pi(X)}\,, \tag{5.5}$$

where $\overline{\pi(X)}$ is the Zariski closure of $\pi(X)$ in $Q_{m,n}$. Now suppose $f \in \mathcal{I}_T(X) \cap C_{m,n}$. Our goal is to show that $f \in J \cap C_{m,n}$. Viewing $f$ as an element of $C_{m,n}$, i.e., a regular function on $Q_{m,n}$, we see that $f \equiv 0$ on $\pi(X)$ and hence, on $\overline{\pi(X)}$. Now applying the usual (commutative) Nullstellensatz to the prime ideal $J \cap C_{m,n}$ of $C_{m,n}$, we see that (5.5) implies $f \in J \cap C_{m,n}$, as desired. □

5.6. **Remark.** In this paper, we consider zeros of ideals of $G_{m,n}$ in $U_{m,n}$. In contrast, Amitsur's Nullstellensatz [1] (see also [2]) deals with zeros in the larger space $(\mathrm{M}_n)^m$. Given an ideal $J$ of $G_{m,n}$, denote by $\mathcal{Z}(J; (\mathrm{M}_n)^m)$ the set of zeroes of $J$ in $(\mathrm{M}_n)^m$. Since $\mathcal{Z}(J; (\mathrm{M}_n)^m) \supset \mathcal{Z}(J)$, it easily follows that

$$J \subset \mathcal{I}\big(\mathcal{Z}(J; (\mathrm{M}_n)^m)\big) \subset \mathcal{I}\big(\mathcal{Z}(J)\big)\,.$$

One particular consequence of Amitsur's Nullstellensatz is that the first inclusion is an equality if $J$ is a prime ideal. Proposition 5.3 implies that both inclusions are equalities, provided $J$ is a prime ideal of PI-degree $n$. Note that the second inclusion can be strict, e.g., if $J$ is a prime ideal of PI-degree $< n$ (since then $\mathcal{Z}(J) = \emptyset$ by Proposition 5.1).

The following theorem summarizes many of our results so far.

5.7. **Theorem.** *Let $n \geq 1$ and $m \geq 2$ be integers.*

(a) *$\mathcal{Z}(\_)$ and $\mathcal{I}(\_)$ are mutually inverse inclusion-reversing bijections between $\mathrm{Spec}_n(G_{m,n})$ and the set of irreducible $n$-varieties $X \subset U_{m,n}$.*

(b) *$\mathcal{Z}(\_)$ and $\mathcal{I}_T(\_)$ are mutually inverse inclusion-reversing bijections between $\mathrm{Spec}_n(T_{m,n})$ and the set of irreducible $n$-varieties $X \subset U_{m,n}$.* □

## 6. Regular maps of $n$-varieties

Recall that an element $g$ of $G_{m,n}$ may be viewed as a regular $\mathrm{PGL}_n$-equivariant map $g\colon (\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$. Now suppose $X$ is an $n$-variety in $U_{m,n}$. Then, restricting $g$ to $X$, we see that $g|_X = g'|_X$ if and only if

$g' - g \in \mathcal{I}(X)$. Hence, elements of the PI-coordinate ring of $X$ may be viewed as $\mathrm{PGL}_n$-equivariant morphisms $X \longrightarrow \mathrm{M}_n$. All of this is completely analogous to the commutative case, where $n = 1$, $\mathrm{M}_1 = k$, $k_1[X] = k[X]$ is the usual coordinate ring of $X \subset k^m$, and elements of $k[X]$ are the regular functions on $X$. It is thus natural to think of elements of $k_n[X]$ as "regular functions" on $X$, even though we shall not use this terminology. (In algebraic geometry, functions are usually assumed to take values in the base field $k$, while elements of $k_n[X]$ take values in $\mathrm{M}_n$.) We also remark that not every $\mathrm{PGL}_n$-equivariant morphism $X \longrightarrow \mathrm{M}_n$ of algebraic varieties (in the usual sense) is induced by elements of $k_n[X]$, see Example 3.3.

6.1. **Definition.** Let $X \subset U_{m,n}$ and $Y \subset U_{l,n}$ be $n$-varieties.

(a) A map $f\colon X \longrightarrow Y$ is called a *regular map* of $n$-varieties, if it is of the form $f = (f_1, \dots, f_l)$ with each $f_i \in k_n[X]$ (so that $f$ sends $a = (a_1, \dots, a_m) \in X \subset (\mathrm{M}_n)^m$ to $(f_1(a), \dots, f_l(a)) \in Y \subset (\mathrm{M}_n)^l$). Note that a morphism of $n$-varieties $X \longrightarrow Y$ extends to a $\mathrm{PGL}_n$-equivariant morphism $(\mathrm{M}_n)^m \longrightarrow (\mathrm{M}_n)^l$.

(b) The $n$-varieties $X$ and $Y$ are called *isomorphic* if there are mutually inverse regular maps $X \longrightarrow Y$ and $Y \longrightarrow X$ .

(c) A regular map $f = (f_1, \dots, f_l)\colon X \longrightarrow Y \subset U_{l,n}$ of $n$-varieties induces a $k$-algebra homomorphism $f^*\colon k_n[Y] \longrightarrow k_n[X]$ given by $\overline{X_i} \longrightarrow f_i$ for $i = 1, \dots, l$, where $\overline{X_1}, \dots, \overline{X_l}$ are the images of the generic matrices $X_1, \dots, X_l \in G_{l,n}$ in $k_n[Y] = G_{l,n}/\mathcal{I}(Y)$. One easily verifies that for every $g \in k_n[Y]$, $f^*(g) = g \circ f\colon X \longrightarrow Y \longrightarrow \mathrm{M}_n$.

(d) Conversely, a $k$-algebra homomorphism $\alpha\colon k_n[Y] \longrightarrow k_n[X]$ induces a regular map $\alpha_* = (f_1, \dots, f_l)\colon X \longrightarrow Y$ of $n$-varieties, where $f_i = \alpha(\overline{X_i})$. It is easy to check that for every $g \in k_n[Y]$, $\alpha(g) = g \circ \alpha_*\colon X \longrightarrow Y \longrightarrow \mathrm{M}_n$.

6.2. **Remark.** It is immediate from these definitions that $(f^*)_* = f$ for any regular map $f\colon X \longrightarrow Y$ of $n$-varieties, and $(\alpha_*)^* = \alpha$ for any $k$-algebra homomorphism $\alpha\colon k_n[Y] \longrightarrow k_n[X]$. Note also that $(\mathrm{id}_X)^* = \mathrm{id}_{k_n[X]}$, and $(\mathrm{id}_{k_n[X]})_* = \mathrm{id}_X$.

6.3. **Lemma.** *Let $X \subset U_{m,n}$ and $Y \subset U_{l,n}$ be $n$-varieties, and let $k_n[X]$ and $k_n[Y]$ be their respective PI-coordinate rings.*

(a) *If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are regular maps of $n$-varieties, then $(g \circ f)^* = f^* \circ g^*$.*
(b) *If $\alpha\colon k_n[Y] \longrightarrow k_n[X]$ and $\beta\colon k_n[Z] \longrightarrow k_n[Y]$ are $k$-algebra homomorphisms, then $(\alpha \circ \beta)_* = \beta_* \circ \alpha_*$.*
(c) *$X$ and $Y$ are isomorphic as $n$-varieties if and only if $k_n[X]$ and $k_n[Y]$ are isomorphic as $k$-algebras.*

*Proof.* Parts (a) and (b) follow directly from Definition 6.1. The proofs are exactly the same as in the commutative case (where $n = 1$); we leave them as an exercise for the reader.

To prove (c), suppose $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ are mutually inverse morphisms of $n$-varieties. Then by part (a), $f^*\colon k_n[Y] \longrightarrow k_n[X]$ and $g^*\colon k_n[X] \longrightarrow k_n[Y]$ are mutually inverse $k$-algebra homomorphisms, showing that $k_n[X]$ and $k_n[Y]$ are isomorphic.

Conversely, if $\alpha\colon k_n[Y] \longrightarrow k_n[X]$ and $\beta\colon k_n[X] \longrightarrow k_n[Y]$ are mutually inverse homomorphisms of $k$-algebras then by part (b), $\alpha_*$ and $\beta_*$ are mutually inverse morphisms between the $n$-varieties $X$ and $Y$. □

6.4. **Theorem.** *Let $R$ be a finitely generated prime $k$-algebra of PI-degree $n$. Then $R$ is isomorphic (as a $k$-algebra) to $k_n[X]$ for some irreducible $n$-variety $X$. Moreover, $X$ is uniquely determined by $R$, up to isomorphism of $n$-varieties.*

*Proof.* By our assumptions on $R$ there exists a surjective ring homomorphism $\varphi\colon G_{m,n} \longrightarrow R$. Then $J_0 = \mathrm{Ker}(\varphi)$ lies in $\mathrm{Spec}_n(G_{m,n})$. Set $X = \mathcal{Z}(J_0) \subset U_{m,n}$. Then $X$ is irreducible (see Corollary 4.3), and $J_0 = \mathcal{I}(X)$ (see Proposition 5.3(a)). Hence $R$ is isomorphic to $G_{m,n}/\mathcal{I}(X) = k_n[X]$, as claimed. The uniqueness of $X$ follows from Lemma 6.3(c). □

We are now ready to prove Theorem 1.1. Recall that for $n = 1$, Theorem 1.1 reduces to [13, Corollary 3.8]. Both are proved by the same argument. Since the proof of [13, Corollary 3.8] is omitted in [13], we reproduce this argument here for the sake of completeness.

*Proof of Theorem 1.1.* By Lemma 6.3, the contravariant functor $\mathcal{F}$ in Theorem 1.1 is well-defined. It is full and faithful by Remark 6.2. Moreover, by Theorem 6.4, every object in $PI_n$ is isomorphic to the image of an object in $Var_n$. Hence $\mathcal{F}$ is a covariant equivalence of categories between $Var_n$ and the dual category of $PI_n$, see, e.g., [8, Theorem 7.6]. In other words, $\mathcal{F}$ is a contravariant equivalence of categories between $Var_n$ and $PI_n$, as claimed. □

We conclude our discussion of regular maps of $n$-varieties with an observation which we will need in the next section.

6.5. **Lemma.** *Let $X$ be an irreducible $n$-variety, and $c$ a central element of $k_n[X]$ or of its trace ring $T_{m,n}/\mathcal{I}_T(X)$. Then the image of $c$ in $\mathrm{M}_n$ consists of scalar matrices.*

*Proof.* Denote by $k_n(X)$ the common total ring of fractions of $k_n[X]$ and $T_{m,n}/\mathcal{I}_T(X)$. By Lemma 2.10, there exists a central polynomial $s \in G_{m,n}$ for $n \times n$ matrices which does not identically vanish on $X$. Then $R = G_{m,n}[s^{-1}]$ is an Azumaya algebra. Consider the natural map $\phi\colon R \longrightarrow k_n(X)$. Then the center of $\phi(R)$ is $\phi(\mathrm{Center}(R))$; see, e.g., [10, Proposition 1.11]. Note that $k_n(X)$ is a central localization of $\phi(R)$. Hence the central element $c$ of $k_n(X)$ is of the form $c = \phi(p)\phi(q)^{-1}$ for central elements $p, q \in G_{m,n}$ with $q \not\equiv 0$ on $X$. All images of $p$ and $q$ in $\mathrm{M}_n$ are central, i.e., scalar matrices. Thus $c(x)$ is a scalar matrix for each $x$ in the dense open subset of $X$ on which $q$ is nonzero. Consequently, $c(x)$ is a scalar matrix for every $x \in X$. □

## 7. Rational maps of $n$-varieties

7.1. **Definition.** Let $X$ be an irreducible $n$-variety. The total ring of fractions of the prime algebra $k_n[X]$ will be called the *central simple algebra of rational functions on* $X$ and denoted by $k_n(X)$.

7.2. **Remark.** One can also define $k_n(X)$ using the trace ring instead of the generic matrix ring. That is, $k_n(X)$ is also the total ring of fractions of $T_{m,n}/\mathcal{I}_T(X)$. Indeed, by Corollary 5.2, $T_{m,n}/\mathcal{I}_T(X)$ is the trace ring of $k_n[X]$, so the two have the same total ring of fractions.

Recall that $k_n(X)$ is obtained from $k_n[X]$ by inverting all non-zero central elements; see, e.g., [26, Theorem 1.7.9]. In other words, every $f \in k_n(X)$ can be written as $f = c^{-1}p$, where $p \in k_n[X]$ and $c$ is a nonzero central element of $k_n[X]$. Recall from Lemma 6.5 that for each $x \in X$, $c(x)$ is a scalar matrix in $\mathrm{M}_n$, and thus invertible if it is nonzero. Viewing $p$ and $c$ as $\mathrm{PGL}_n$-equivariant morphisms $X \longrightarrow \mathrm{M}_n$ (in the usual sense of commutative algebraic geometry), we see that $f$ can be identified with a rational map $c^{-1}p \colon X \dashrightarrow \mathrm{M}_n$. One easily checks that this map is independent of the choice of $c$ and $p$, i.e., remains the same if we replace $c$ and $p$ by $d$ and $q$, such that $f = c^{-1}p = d^{-1}q$. We will now see that every $\mathrm{PGL}_n$-equivariant rational map $X \dashrightarrow \mathrm{M}_n$ is of this form.

7.3. **Proposition.** *Let $X \subset U_{m,n}$ be an irreducible $n$-variety. Then the natural inclusion $k_n(X) \hookrightarrow \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is an isomorphism.*

Here $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ denotes the $k$-algebra of $\mathrm{PGL}_n$-equivariant rational maps $X \dashrightarrow \mathrm{M}_n$, with addition and multiplication induced from $\mathrm{M}_n$. Recall that a regular analogue of Proposition 7.3 (with rational maps replaced by regular maps, and $k_n(X)$ replaced by $k_n[X]$) is false; see Remark 3.2 and Example 3.3.

*First proof* (*algebraic*). Recall that $k_n(X)$ is, by definition, a central simple algebra of PI-degree $n$. By [24, Lemma 8.5] (see also [24, Definition 7.3 and Lemma 9.1]), $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is a central simple algebra of PI-degree $n$ as well. It is thus enough to show that the centers of $k_n(X)$ and $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ coincide.

Let $\overline{X}$ be the closure of $X$ in $(\mathrm{M}_n)^m$. By [24, Lemma 8.5], the center of $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n) = \mathrm{RMaps}_{\mathrm{PGL}_n}(\overline{X}, \mathrm{M}_n)$ is the field $k(X)^{\mathrm{PGL}_n}$ of $\mathrm{PGL}_n$-invariant rational functions $f \colon X \dashrightarrow k$ (or equivalently, the field $k(\overline{X})^{\mathrm{PGL}_n}$). Here, as usual, we identify $f$ with $f \cdot I_n \colon X \dashrightarrow \mathrm{M}_n$. It now suffices to show that

$$k(X)^{\mathrm{PGL}_n} = \mathrm{Center}(k_n(X))\,. \tag{7.4}$$

Recall from Lemma 2.6(a) that the natural algebra homomorphism $G_{m,n} \longrightarrow k_n(X)$ extends to a homomorphism $T_{m,n} \longrightarrow k_n(X)$. So the center of

$k_n(X)$ contains all functions $f|_X \colon X \longrightarrow k$, as $f$ ranges over the ring $C_{m,n} = k[(\mathrm{M}_n)^m]^{\mathrm{PGL}_n}$. Since $X \subset U_{m,n}$, these functions separate the $\mathrm{PGL}_n$-orbits in $X$; see Proposition 2.3(b). Equality (7.4) now follows a theorem of Rosenlicht; cf. [18, Lemma 2.1]. □

*Alternative proof* (*geometric*). By Remark 7.2, it suffices to show that for every $\mathrm{PGL}_n$-equivariant rational map $f \colon X \dashrightarrow \mathrm{M}_n$ there exists an $h \in k[\mathrm{M}_n]^{\mathrm{PGL}_n} = C_{m,n}$ such that $h \not\equiv 0$ on $X$ and $hf \colon x \mapsto h(x)f(x)$ lifts to a regular map $(\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$ (and in particular, $hf$ is a regular map $X \longrightarrow \mathrm{M}_n$).

It is enough to show that the ideal $I \subset k[(\mathrm{M}_n)^m]$ given by

$$I = \{h \in k[(\mathrm{M}_n)^m] \mid hf \text{ lifts to a regular map } (\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n\}$$

contains a $\mathrm{PGL}_n$-invariant element $h$ such that $h \not\equiv 0$ on $X$. Indeed, regular $\mathrm{PGL}_n$-equivariant morphisms $(\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$ are precisely elements of $T_{m,n}$; hence, $hf \colon X \longrightarrow \mathrm{M}_n$ would then lie in $k_n(X)$, and so would $f = h^{-1}(hf)$, thus proving the lemma.

Denote by $Z$ the zero locus of $I$ in $(\mathrm{M}_n)^m$ (in the usual sense, not in the sense of Definition 3.1(b)). Then $Z \cap X$ is, by definition the indeterminacy locus of $f$; in particular, $X \not\subset Z$. Choose $a \in X \setminus Z$ and let $C = \mathrm{PGL}_n \cdot a$ be the orbit of $a$ in $X$. Since $a \in X \subset U_{m,n}$, Proposition 2.3(b) tells us that $C$ is closed in $(\mathrm{M}_n)^m$. In summary, $C$ and $Z$ are disjoint $\mathrm{PGL}_n$-invariant Zariski closed subsets of $(\mathrm{M}_n)^m$. Since $\mathrm{PGL}_n$ is reductive, they can be separated by a regular invariant, i.e., there exists a $0 \neq j \in k[(\mathrm{M}_n)^m]^{\mathrm{PGL}_n}$ such that $j(a) \neq 0$ but $j \equiv 0$ on $Z$; see, e.g., [16, Corollary 1.2]. By Hilbert's Nullstellensatz, $h = j^r$ lies in $I$ for some $r \geq 1$. This $h$ has the desired properties: it is a $\mathrm{PGL}_n$-invariant element of $I$ which is not identically zero on $X$. □

7.5. **Definition.** Let $X \subset U_{m,n}$ and $Y \subset U_{l,n}$ be irreducible $n$-varieties.

(a) A rational map $f \colon X \dashrightarrow Y$ is called a *rational map of $n$-varieties* if $f = (f_1, \dots, f_l)$ where each $f_i \in k_n(X)$. Equivalently (in view of Proposition 7.3), a rational map $X \dashrightarrow Y$ of $n$-varieties is simply a $\mathrm{PGL}_n$-equivariant rational map (in the usual sense).

(b) The $n$-varieties $X$ and $Y$ are called *birationally isomorphic* or *birationally equivalent* if there exist dominant rational maps of $n$-varieties $f \colon X \dashrightarrow Y$ and $g \colon Y \dashrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ (as rational maps of varieties).

(c) A dominant rational map $f = (f_1, \dots, f_l) \colon X \dashrightarrow Y$ of $n$-varieties induces a $k$-algebra homomorphism (i.e., an embedding) $f^* \colon k_n(Y) \longrightarrow k_n(X)$ of central simple algebras defined by $f^*(\overline{X_i}) = f_i$, where $\overline{X_i}$ is the image of the generic matrix $X_i \in G_{l,n}$ in $k_n[Y] \subset k_n(Y)$. One easily verifies that for every $g \in k_n(Y)$, $f^*(g) = g \circ f$, if one views $g$ as a $\mathrm{PGL}_n$-equivariant rational map $Y \dashrightarrow \mathrm{M}_n$.

(d) Conversely, a $k$-algebra homomorphism (necessarily an embedding) of central simple algebras $\alpha\colon k_n(Y) \longrightarrow k_n(X)$ (over $k$) induces a dominant rational map $f = \alpha_*\colon X \dashrightarrow Y$ of $n$-varieties. This map is given by $f = (f_1, \dots, f_l)$ with $f_i = \alpha(\overline{X_i}) \in k_n(X)$, where $\overline{X_1}, \dots, \overline{X_l}$ are the images of the generic matrices $X_1, \dots, X_l \in G_{l,n}$. It is easy to check that for every $g \in k_n(Y)$, $\alpha(g) = g \circ \alpha_*$, if one views $g$ as a $\mathrm{PGL}_n$-equivariant rational map $Y \dashrightarrow \mathrm{M}_n$.

7.6. **Remark.** Once again, the identities $(f^*)_* = f$ and $(\alpha_*)^* = \alpha$ follow directly from these definitions. Similarly, $(\mathrm{id}_X)^* = \mathrm{id}_{k_n(X)}$ and $(\mathrm{id}_{k_n(X)})_* = \mathrm{id}_X$.

We also have the following analogue of Lemma 6.3 for dominant rational maps. The proofs are again the same as in the commutative case (where $n = 1$); we leave them as an exercise for the reader.

7.7. **Lemma.** *Let $X \subset U_{m_1,n}$, $Y \subset U_{m_2,n}$ and $Z \subset U_{m_3,n}$ be irreducible $n$-varieties.*

(a) *If $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ are dominant rational maps of $n$-varieties then $(g \circ f)^* = f^* \circ g^*$.*
(b) *If $\alpha\colon k_n(Y) \hookrightarrow k_n(X)$ and $\beta\colon k_n(Z) \hookrightarrow k_n(Y)$ are homomorphisms (i.e., embeddings) of central simple algebras then $(\alpha \circ \beta)_* = \beta_* \circ \alpha_*$.*
(c) *$X$ and $Y$ are birationally isomorphic as $n$-varieties if and only if the central simple algebras $k_n(X)$ and $k_n(Y)$ are isomorphic as $k$-algebras.* □

We are now ready to prove the following birational analogue of Theorem 6.4.

7.8. **Theorem.** *Let $K/k$ be a finitely generated field extension and $A$ be a central simple algebra of degree $n$ with center $K$. Then $A$ is isomorphic (as a $k$-algebra) to $k_n(X)$ for some irreducible $n$-variety $X$. Moreover, $X$ is uniquely determined by $A$, up to birational isomorphism of $n$-varieties.*

*Proof.* Choose generators $a_1, \dots, a_N \in K$ for the field extension $K/k$ and a $K$-vector space basis $b_1, \dots, b_{n^2}$ for $A$. Let $R$ be the $k$-subalgebra of $A$ generated by all $a_i$ and $b_j$. By our construction $R$ is a prime $k$-algebra of PI-degree $n$, with total ring of fraction $A$. By Theorem 6.4 there exists an $n$-variety $X$ such that $k_n[X] \simeq R$ and hence, $k_n(X) \simeq A$. This proves the existence of $X$. Uniqueness follows from Lemma 7.7(c). □

## 8. Generically free $\mathrm{PGL}_n$-varieties

An irreducible $n$-variety is, by definition, an irreducible generically free $\mathrm{PGL}_n$-variety. The following lemma says that up to birational isomorphism, the converse is true as well.

8.1. **Lemma.** *Every irreducible generically free $\mathrm{PGL}_n$-variety $X$ is birationally isomorphic (as $\mathrm{PGL}_n$-variety) to an irreducible $n$-variety in $U_{m,n}$ for some $m \geq 2$.*

*Proof.* Choose $a \in U_{2,n}$. By [24, Proposition 7.1] there exists a $\mathrm{PGL}_n$-equivariant rational map $\phi\colon X \dashrightarrow (\mathrm{M}_n)^2$ whose image contains $a$. Now choose $\mathrm{PGL}_n$-invariant rational functions $c_1, \dots, c_r \in k(X)^{\mathrm{PGL}_n}$ on $X$ which separate $\mathrm{PGL}_n$-orbits in general position (this can be done by a theorem of Rosenlicht; cf. e.g., [18, Theorem 2.3]). We now set $m = r + 2$ and define $f\colon X \dashrightarrow (\mathrm{M}_n)^m$ by

$$f(x) = \big(c_1(x)I_{n\times n}, \dots, c_r(x)I_{n\times n}, \phi(x)\big) \in (\mathrm{M}_n)^r \times (\mathrm{M}_n)^2 = (\mathrm{M}_n)^m .$$

Let $\overline{Y}$ be the Zariski closure of $f(X)$ in $(\mathrm{M}_n)^m$, and $Y = \overline{Y} \cap U_{m,n}$. By our choice of $\phi$, $Y \neq \emptyset$. It thus remains to be shown that $f$ is a birational isomorphism between $X$ and $Y$ (or, equivalently, $\overline{Y}$). Since we are working over a base field $k$ of characteristic zero, it is enough to show that $X$ has a dense open subset $S$ such that $f(a) \neq f(b)$ for every pair of distinct $k$-points $a, b \in S$.

Indeed, choose $S \subset X$ so that (i) the generators $c_1, \dots, c_r$ of $k(X)^{\mathrm{PGL}_n}$ separate $\mathrm{PGL}_n$-orbits in $S$, (ii) $f$ is well-defined in $S$ and (iii) $f(S) \subset U_{m,n}$. Now let $a$, $b \in S$, and assume that $f(a) = f(b)$. Then $a$ and $b$ must belong to the same $\mathrm{PGL}_n$-orbit. Say $b = h(a)$, for some $h \in \mathrm{PGL}_n$. Then $f(a) = f(b) = hf(a)h^{-1}$. Since $f(a) \in U_{m,n}$, $h = 1$, so that $a = b$, as claimed. □

Lemma 8.1 suggests that in the birational setting the natural objects to consider are arbitrary generically free $\mathrm{PGL}_n$-varieties, rather than $n$-varieties. The relationship between the two is analogous to the relationship between affine varieties and more general algebraic (say, quasi-projective) varieties in the usual setting of (commutative) algebraic geometry. In particular, in general one cannot assign a PI-coordinate ring $k_n[X]$ to an irreducible generically free $\mathrm{PGL}_n$-variety in a meaningful way. On the other hand, we can extend the definition of $k_n(X)$ to this setting as follows.

8.2. **Definition.** Let $X$ be an irreducible generically free $\mathrm{PGL}_n$-variety. Then $k_n(X)$ is the $k$-algebra of $\mathrm{PGL}_n$-equivariant rational maps $f\colon X \dashrightarrow \mathrm{M}_n$, with addition and multiplication induced from $\mathrm{M}_n$.

Proposition 7.3 tells us that if $X$ is an irreducible $n$-variety then this definition is consistent with Definition 7.1. In place of $k_n(X)$ we will sometimes write $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$.

8.3. **Definition.** A dominant rational map $f\colon X \dashrightarrow Y$ of generically free $\mathrm{PGL}_n$-varieties gives rise to a homomorphism (embedding) $f^*\colon k_n(Y) \longrightarrow k_n(X)$ given by $f^*(g) = g \circ f$ for every $g \in k_n(Y)$.

If $X$ and $Y$ are $n$-varieties, this definition of $f^*$ coincides with Definition 7.5(c). Note that $(\mathrm{id}_X)^* = \mathrm{id}_{k_n(X)}$, and that $(g \circ f)^* = f^* \circ g^*$ if $g\colon Y \dashrightarrow Z$ is another $\mathrm{PGL}_n$-equivariant dominant rational map. We will now show that Definition 7.5 and Remark 7.6 extend to this setting as well.

8.4. **Proposition.** *Let $X$ and $Y$ be generically free irreducible* $\mathrm{PGL}_n$*-varieties and*

$$\alpha\colon k_n(X) \to k_n(Y)$$

*be a $k$-algebra homomorphism. Then there is a unique* $\mathrm{PGL}_n$*-equivariant, dominant rational map $\alpha_*\colon Y \dashrightarrow X$ such that $(\alpha_*)^* = \alpha$.*

*Proof.* If $X$ and $Y$ are $n$-varieties, i.e., closed $\mathrm{PGL}_n$-invariant subvarieties of $U_{m,n}$ and $U_{l,n}$ respectively (for some $m, l \geq 2$) then $\alpha_*\colon Y \dashrightarrow X$ is given by Definition 7.5(d), and uniqueness follows from Remark 7.6.

In general, Lemma 8.1 tells us that there are birational isomorphisms $X \dashrightarrow X'$ and $Y \dashrightarrow Y'$ where $X'$ and $Y'$ are $n$-varieties. The proposition is now a consequence of the following lemma. □

8.5. **Lemma.** *Let $f\colon X \dashrightarrow X'$ and $g\colon Y \dashrightarrow Y'$ be birational isomorphisms of* $\mathrm{PGL}_n$*-varieties. If Proposition 8.4 holds for $X'$ and $Y'$ then it holds for $X$ and $Y$.*

*Proof.* Note that by our assumption, the algebra homomorphism

$$\beta = (g^*)^{-1} \circ \alpha \circ f^*\colon k_n(X') \to k_n(Y')$$

is induced by the $\mathrm{PGL}_n$-equivariant, dominant rational map $\beta_*\colon Y' \to X'$:

$$\begin{array}{ccc} k_n(X) & \xrightarrow{\alpha} & k_n(Y) \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle g^*} \\ k_n(X') & \xrightarrow{\beta} & k_n(Y') \end{array} \qquad\qquad \begin{array}{ccc} Y & & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \overset{\beta_*}{\dashrightarrow} & X' \end{array}$$

Now one easily checks that the dominant rational map

$$\alpha_* := f^{-1} \circ \beta_* \circ g\colon Y \dashrightarrow X$$

has the desired property: $(\alpha_*)^* = \alpha$. This shows that $\alpha_*$ exists. To prove uniqueness, let $h\colon Y \longrightarrow X$ be another $\mathrm{PGL}_n$-equivariant dominant rational map such that $h^* = \alpha$. Then $(f \circ h \circ g^{-1})^* = (g^{-1})^* \circ \alpha \circ f^* = \beta$. By uniqueness of $\beta_*$, we have $f \circ h \circ g^{-1} = \beta_*$, i.e., $h = f^{-1} \circ \beta_* \circ g = \alpha_*$. This completes the proof of Lemma 8.5 and thus of Proposition 8.4. □

8.6. **Corollary.** *Let $X$, $Y$ and $Z$ be generically free irreducible* $\mathrm{PGL}_n$*-varieties.*

(a) *If $f\colon X \dashrightarrow Y$, $g\colon Y \dashrightarrow Z$ are* $\mathrm{PGL}_n$*-equivariant dominant rational maps then $(g \circ f)^* = f^* \circ g^*$.*
(b) *If $\alpha\colon k_n(Y) \hookrightarrow k_n(X)$ and $\beta\colon k_n(Z) \hookrightarrow k_n(Y)$ are homomorphisms (i.e., embeddings) of central simple algebras then $(\alpha \circ \beta)_* = \beta_* \circ \alpha_*$.*
(c) *$X$ and $Y$ are birationally isomorphic as* $\mathrm{PGL}_n$*-varieties if and only if $k_n(X)$ and $k_n(Y)$ are isomorphic as $k$-algebras.*

*Proof.* (a) is immediate from Definition 8.3.

(b) Let $f = (\alpha \circ \beta)_*$ and $g = \beta_* \circ \alpha_*$. Part (a) tells us that $f^* = g^*$. The uniqueness assertion of Proposition 8.4 now implies $f = g$.

(c) follows from (a) and (b) and the identities $(\mathrm{id}_X)^* = \mathrm{id}_{k_n(X)}$, and $(\mathrm{id}_{k_n(X)})_* = \mathrm{id}_X$. $\square$

*Proof of Theorem 1.2.* We use the same argument as in the proof of Theorem 1.1. The contravariant functor $\mathcal{F}$ is well defined by Corollary 8.6. Since $(f^*)_* = f$ and $(\alpha_*)^* = \alpha$ (see Proposition 8.4), $\mathcal{F}$ is full and faithful. By Theorem 7.8, every object in $CS_n$ is isomorphic to the image of an object in $Bir_n$. The desired conclusion now follows from [8, Theorem 7.6]. $\square$

## 9. Brauer-Severi Varieties

Let $K/k$ be a finitely generated field extension. Recall that the following sets are in a natural (i.e., functorial in $K$) bijective correspondence:

(1) the Galois cohomology set $H^1(K, \mathrm{PGL}_n)$,
(2) central simple algebras $A$ of degree $n$ with center $K$,
(3) Brauer-Severi varieties over $K$ of dimension $n-1$,
(4) $\mathrm{PGL}_n$-torsors over $\mathrm{Spec}(K)$,
(5) pairs $(X, \phi)$, where is $X$ is an irreducible generically free $\mathrm{PGL}_n$-variety and $\phi_X : k(X)^{\mathrm{PGL}_n} \xrightarrow{\simeq} K$ is an isomorphism of fields (over $k$). Two such pairs $(X, \phi)$ and $(Y, \psi)$ are equivalent, if there is a $\mathrm{PGL}_n$-equivariant birational isomorphism $f : Y \dashrightarrow X$ which is compatible with $\phi$ and $\psi$, i.e., there is a commutative diagram

$$\begin{array}{ccc} k(X)^{\mathrm{PGL}_n} & \xrightarrow{f^*} & k(Y)^{\mathrm{PGL}_n} \\ & {\scriptstyle \phi,\ \simeq} \searrow \quad \swarrow {\scriptstyle \psi,\ \simeq} & \\ & K & \end{array}$$

Bijective correspondences between (1), (2), (3) and (4) follow from the theory of descent; see [29, Sections I.5 and III.1], [30, Chapter X], [6, (1.4)] or [14, Sections 28, 29]. For a bijective correspondence between (1) and (5), see [17, (1.3)].

For notational simplicity we will talk of generically free $\mathrm{PGL}_n$-varieties $X$ instead of pairs $(X, \phi)$ in (5), and we will write $k(X)^{\mathrm{PGL}_n} = K$ instead of $k(X)^{\mathrm{PGL}_n} \overset{\phi}{\simeq} K$, keeping $\phi$ in the background.

Suppose we are given a generically free $\mathrm{PGL}_n$-variety $X$ (as in (5)). Then this variety defines a class $\alpha \in H^1(K, \mathrm{PGL}_n)$ (as in (1)) and using this class we can recover all the other associated objects (2) - (4). In the previous section we saw that the central simple algebra $A$ can be constructed directly from $X$, as $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. The goal of this section is to describe a way to pass directly from (5) to (3), without going through (1); this is done in Proposition 9.2 below.

In order to state Proposition 9.2, we introduce some notation. We will write points of the projective space $\mathbb{P}^{n-1} = \mathbb{P}^{n-1}_k$ as rows $a = (a_1 : \ldots : a_n)$.

The group $\mathrm{PGL}_n$ acts on $\mathbb{P}^{n-1}$ by multiplication on the right:

$$g\colon (a_1 : \ldots : a_n) \mapsto (a_1 : \ldots : a_n)g^{-1}\,.$$

Choose (and fix) $a = (a_1 : \ldots : a_n) \in \mathbb{P}^{n-1}$ and define the maximal parabolic subgroup $H$ of $\mathrm{PGL}_n$ by

$$H = \{h \in \mathrm{PGL}_n \mid ah^{-1} = a\}\,. \tag{9.1}$$

If $a = (1 : 0 : \cdots : 0)$ then $H \subset \mathrm{PGL}_n$ consists of of $n \times n$-matrices of the form

$$\begin{pmatrix} * & 0 & \ldots & 0 \\ * & * & \ldots & * \\ \vdots & \vdots & & \vdots \\ * & * & \ldots & * \end{pmatrix}$$

9.2. **Proposition.** *Let $X$ be an irreducible generically free $\mathrm{PGL}_n$-variety, $A = k_n(X)$ and $\sigma\colon X/H \dashrightarrow X/\mathrm{PGL}_n$. Then the Brauer-Severi variety $\mathrm{BS}(A)$ is the preimage of the generic point $\eta$ of $X/\mathrm{PGL}_n$ under $\sigma$.*

Before we proceed with the proof, three remarks are in order. First of all, by $X/H$ we mean the rational quotient variety for the $H$-action on $X$. Recall that $X/H$ is defined (up to birational isomorphism) by $k(X/H) = k(X)^H$, and the dominant rational map $\sigma\colon X/H \dashrightarrow X/\mathrm{PGL}_n$ by the inclusion of fields $k(X)^{\mathrm{PGL}_n} \hookrightarrow k(X)^H$. Secondly, recall that $k(X/\mathrm{PGL}_n) = k(X)^{\mathrm{PGL}_n} = K$, so that $\eta \simeq \mathrm{Spec}(K)$, and $\sigma^{-1}(\eta)$ is, indeed, a $K$-variety. Thirdly, while the construction of $\mathrm{BS}(A)$ in Proposition 9.2 does not use the Galois cohomology set $H^1(K, \mathrm{PGL}_n)$, our proof below does. In fact, our argument is based on showing that $\sigma^{-1}(\eta)$ and $\mathrm{BS}(A)$ are Brauer-Severi varieties defined by the same class in $H^1(K, \mathrm{PGL}_n)$.

*Proof.* Let $X_0/k$ be an algebraic variety with function field $k(X_0) = K$, i.e., a particular model for the rational quotient variety $X/\mathrm{PGL}_n$. The inclusion $K \overset{\phi}{\simeq} k(X)^{\mathrm{PGL}_n} \hookrightarrow X$ induces the rational quotient map $\pi\colon X \dashrightarrow X_0$. After replacing $X_0$ by a Zariski dense open subset, we may assume $\pi$ is regular; after passing to another (smaller) dense open subset, we may assume $\pi\colon X \longrightarrow X_0$ is, in fact, a torsor; cf., e.g., [18, Section 2.5].

We now trivialize this torsor over some etale cover $U_i \to X_0$. Then for each $i, j$ the transition map $f_{ij}\colon \mathrm{PGL}_n \times U_{ij} \longrightarrow \mathrm{PGL}_n \times U_{ij}$ is an automorphism of the trivial $\mathrm{PGL}_n$-torsor $\mathrm{PGL}_n \times U_{ij}$ on $U_{ij}$. It is easy to see that $f_{ij}$ is given by the formula

$$f_{ij}(u, g) = (u, g \cdot c_{ij}(u))\,, \tag{9.3}$$

for some morphism $c_{ij}\colon U_{ij} \longrightarrow \mathrm{PGL}_n$. The morphisms $c_{ij}$ satisfy a cocycle condition (for Cech cohomology) which expresses the fact that the transition maps $f_{ij}$ are compatible on triple "overlaps" $U_{hij}$. The cocycle $c = (c_{ij})$ gives rise to a cohomology class $\overline{c} \in H^1(X_0, \mathrm{PGL}_n)$, which maps to $\alpha$ under the natural restriction morphism $H^1(X_0, \mathrm{PGL}_n) \longrightarrow H^1(K, \mathrm{PGL}_n)$ from $X_0$ to its generic point; cf. [9, Section 8]. (Recall that by our construction the function field of $X_0$ is identified with $K$.)

Now define the quotient $Z$ of $X$ by the maximal parabolic subgroup $H \subset \mathrm{PGL}_n$ as follows. Over each $U_i$ set $Z_i = H\backslash \mathrm{PGL}_n \times U_i$. By descent we can "glue" the projection morphisms $Z_i \longrightarrow U_i$ into a morphism $Z \longrightarrow X_0$ by the transition maps

$$\overline{f_{ij}}(u, g) = (u, \overline{g} \cdot c_{ij}(u)) .$$

Moreover, since over each $U_i$ the map $\pi\colon X \longrightarrow X_0$ factors as

$$\pi_i\colon \mathrm{PGL}_n \times U_i \xrightarrow{p_i} H\backslash \mathrm{PGL}_n \times U_i \xrightarrow{q_i} U_i ,$$

the projection maps $p_i$ and $q_i$ also glue together, yielding

$$\pi\colon X \xrightarrow{p} Z \xrightarrow{q} X_0 .$$

By our construction the fibers of $p$ are exactly the $H$-orbits in $X$; hence, $k(Z) = k(X)^H$, cf., e.g., [18, 2.1]. In other words, $p$ is a rational quotient map for the $H$-action on $X$ and we can identify $Z$ with the rational quotient variety $X/H$ (up to birational equivalence). Under this identification $q$ becomes $\sigma$.

Now recall that by the definition of $H$, the homogeneous space $H\backslash \mathrm{PGL}_n$ is naturally isomorphic with $\mathbb{P}^{n-1}$ via $\overline{g} \mapsto g \cdot a$. Since over each $U_i$ the map $q\colon Z \longrightarrow X_0$ looks like the projection $H\backslash \mathrm{PGL}_n \times U_i \longrightarrow U_i$, $Z$ is, by definition, a Brauer-Severi variety over $X_0$. Moreover, $\pi\colon X \longrightarrow X_0$ (viewed as a torsor over $X_0$) and $q\colon Z \longrightarrow X_0$ (viewed as a Brauer-Severi variety over $X_0$) are constructed by using the same cocycle $(c_{ij})$ and hence, the same cohomology class $\overline{c} \in H^1(X_0, \mathrm{PGL}_n)$. Restricting to the generic point of $X_0$, we see that the cohomology class of $Z$ as a Brauer-Severi variety over $K = k(X_0)$ is the image of $\overline{c}$ under the restriction map $H^1(X_0, \mathrm{PGL}_n) \longrightarrow H^1(K, \mathrm{PGL}_n)$, i.e., the class $\alpha \in H^1(K, \mathrm{PGL}_n)$ we started out with. □

9.4. **Remark.** Note that the choice of the maximal parabolic subgroup $H \subset \mathrm{PGL}_n$ is important here. If we repeat the same construction with $H$ replaced by $H^{\mathrm{transpose}}$ we will obtain the Brauer-Severi variety of the opposite algebra $A^{\mathrm{op}}$.

The following corollary of Proposition 9.2 shows that $X$, viewed as an abstract variety (i.e., without the $\mathrm{PGL}_n$-action), is closely related to $\mathrm{BS}(A)$.

9.5. **Corollary.** *Let $X$ be a generically free $\mathrm{PGL}_n$-variety, $A = k_n(X)$ be the associated central simple algebra of degree $n$, and $K = k(X)^{\mathrm{PGL}_n}$ be the center of $A$. Then*

$$k(X) \simeq K(\mathrm{BS}(A))(t_1, \dots, t_{n^2-n}) \simeq K(\mathrm{BS}(\mathrm{M}_n(A))) .$$

Here $k(X)$ denotes the function field of $X$ (as a variety over $k$), and $K(\mathrm{BS}(A))$ and $K(\mathrm{BS}(\mathrm{M}_n(A))$ denote, respectively, the function fields of the Brauer-Severi varieties of $A$ and $\mathrm{M}_n(A)$ (both are defined over $K$). The letters $t_1, \dots, t_{n^2-1}$ denote $n^2-n$ independent commuting variables, and the isomorphisms $\simeq$ are field isomorphisms over $k$ (they ignore the $\mathrm{PGL}_n$-action on $X$).

*Proof.* The second isomorphism is due to Roquette [25, Theorem 4, p. 413]. To show that $k(X) \simeq K(\mathrm{BS}(A))(t_1, \dots, t_{n^2-n})$, note that by Proposition 9.2, $K(\mathrm{BS}(A)) = K(X/H) = k(X)^H$, where $H$ is the parabolic subgroup of $\mathrm{PGL}_n$ defined in (9.1). Since $\dim(H) = n^2 - n$, it remains to show that the field extension $k(X)/k(X)^H$ is rational.

Now recall that $H$ is a special group (cf. [18, Section 2.6]); indeed, the Levi subgroup of $H$ is isomorphic to $\mathrm{GL}_{n-1}$. Consequently, $X$ is birationally isomorphic to $(X/H) \times H$ (over $k$). Since $k$ is assumed to be algebraically closed and of characteristic zero, every algebraic group over $k$ is rational. In particular, $H$ is birationally isomorphic to $\mathbb{A}^{n^2-n}$ and thus $X$ is birationally isomorphic to $(X/H) \times \mathbb{A}^{n^2-n}$. In other words,

$$k(X) \simeq K(X/H)(t_1, \dots, t_{n^2-n}) \simeq K(\mathrm{BS}(A))(t_1, \dots, t_{n^2-n}),$$

as claimed. □

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