

Stable Affine Models for Algebraic Group Actions

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Abstract. Let G be a reductive linear algebraic group defined over an algebraically closed base field k of characteristic zero. A G -variety is an algebraic variety with a regular action of G , defined over k . An affine G -variety is called stable if its points in general position have closed G -orbits. We give a simple necessary and sufficient condition for a G -variety to have a stable affine birational model.

1. Introduction

Let G be a linear algebraic group, defined over an algebraically closed base field k of characteristic zero. We shall refer to a reduced but not necessarily irreducible algebraic variety X (defined over k), with a regular action of G (also defined over k) as a G -variety. By a morphism $X \rightarrow Y$ of G -varieties, we shall mean a G -equivariant morphism. The notions of isomorphism, rational map, birational isomorphism, etc. of G -varieties are defined in a similar manner. As usual, given a G -action on X , we shall denote the orbit of $x \in X$ by Gx and the stabilizer subgroup of x by $G_x \subseteq G$. Finally, we shall say that a property holds for $x \in X$ in general position if it holds for every point x of some dense open subset of X .

In this note we will be interested in studying G -varieties up to birational isomorphism. In this context it is natural to ask whether or not a given G -variety X has an affine model. Indeed, there are numerous results and constructions in invariant theory that are available for affine G -varieties but not in general, especially if G is reductive; cf. [5].

Recall that an affine G -variety X is called *stable*, if the orbit Gx is closed for $x \in X$ in general position. If G is reductive, these varieties have many nice properties; for a summary, see, e.g., [10, Section 8]. The question we will address in this note is: Which G -varieties have a stable affine birational model? Our main result is the following:

Theorem 1.1. *Let G be a reductive linear algebraic group and X be a G -variety. Then the following are equivalent:*

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- (a) X is birationally equivalent to a stable affine G -variety.
- (b) The stabilizer G_x is reductive for x in general position in X .

In the case where $X = G/H$ is a homogeneous space, Theorem 1.1 reduces to a theorem of Matsushima [3] which says that G/H is affine if and only if H is reductive. Moreover, the implication (a) \implies (b) of Theorem 1.1 is an immediate consequence of Matsushima's theorem. Indeed, after replacing X by a stable affine model, we see that for $x \in X$ in general position the orbit $Gx \simeq G/G_x$ is affine, so that G_x is reductive.

Our proof of the implication (b) \implies (a) will be based on the following more general result:

Theorem 1.2. *Let G be a linear algebraic group and X a G -variety. Denote by G_x the stabilizer of $x \in X$ in G . Assume that one of the following two conditions is satisfied:*

- (i) $G_x = \{1\}$ for $x \in X$ in general position (i.e., the G -action on X is generically free), or
- (ii) the normalizer $N_G(G_x)$ is reductive for $x \in X$ in general position.

Then X is birationally isomorphic to a stable affine G -variety.

Note that if G and G_x are both reductive then so is the normalizer $N_G(G_x)$; see [2, Lemma 1.1]. Thus Theorem 1.2(ii) proves the implication (b) \implies (a) of Theorem 1.1.

The rest of this note will be devoted to proving Theorem 1.2. Our proof of part (ii) will be based on part (i) and a theorem of Richardson [7, Theorem 9.3.1] about the existence of stabilizers in general position.

We remark that the theorems of Matsushima and Richardson mentioned above were originally proved only for $k = \mathbb{C}$ (by analytic methods). An algebraic proof of Matsushima's theorem over an algebraically closed field k of characteristic zero can be found in [1, Section 2]. Richardson's theorem is also valid over such k by the Lefschetz principle; it is stated in this form in [5, Theorem 7.1]. Nevertheless, it would be interesting to find a direct algebraic proof.

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2. Proof of Theorem 1.2(i)

We begin with a simple lemma.

Lemma 2.1. *Every linear algebraic group G has a stable generically free linear representation.*

Proof. We first embed G as a closed subgroup in SL_n for some $n \geq 1$. The linear action of SL_n on $V = M_n(k)$ by left multiplication is easily seen to be stable and generically free. So for $v \in V$ in general position, the stabilizer of v

in SL_n is trivial, and $\mathrm{SL}_n v$ is closed in V . For such a v , the map $h \mapsto hv$ gives an isomorphism of SL_n onto $\mathrm{SL}_n v$. Hence, Gv is closed in $\mathrm{SL}_n v$ and thus in V . Consequently, the induced linear action of G on V is stable and generically free. ■

We are now ready to proceed with the proof of Theorem 1.2(i). Recall that a G -variety is called *primitive* if G transitively permutes the irreducible components of X . It is easy to see that every X is birationally isomorphic to a disjoint union of primitive G -varieties; cf. [6, Lemma 2.2]. Hence, we may assume that X is primitive.

By a theorem of Rosenlicht there exists a rational quotient map

$$\pi_{\mathrm{rat}}: X \dashrightarrow Z,$$

separating the G -orbits in general position in X ; see [8] (for the case where X is irreducible) and [9] (for general X). Here Z is only defined up to birational isomorphism, so we may assume without loss of generality that Z is affine. After replacing X by a dense open G -invariant subset, we may assume that $G_x = \{1\}$ for every $x \in X$, and that π_{rat} is regular and separates the G -orbits in X . Since X is primitive, Z is irreducible.

By Lemma 2.1 there exists a stable generically free linear representation V of G . Let V_0 be a G -stable dense open subset of V such that every point $v \in V_0$ has a closed orbit (in V) and trivial stabilizer. By [6, Proposition 7.1] there is a G -equivariant rational map $f: X \dashrightarrow V$ whose image contains a point $v \in V_0$. Let Y be the closure of the image of $f \times \pi_{\mathrm{rat}}: X \dashrightarrow V \times Z$. Note that Y is G -primitive and affine. Moreover, $U = Y \cap (V_0 \times Z)$ is a G -invariant non-empty (and hence, dense) open subset of Y , and every point of U has a trivial stabilizer in G and a closed G -orbit in $V \times Z$. Thus Y is a stable affine generically free G -variety.

It remains to show that $f \times \pi_{\mathrm{rat}}$ is a birational isomorphism between X and Y . Since we are working in characteristic zero, since X is primitive, and since $f \times \pi_{\mathrm{rat}}: X \dashrightarrow Y$ is dominant, it suffices to check that $f \times \pi_{\mathrm{rat}}$ is injective on a dense open subset of X . Indeed, let $W = (f \times \pi_{\mathrm{rat}})^{-1}(U)$. Then W is a G -stable nonempty (and thus dense) open subset of X . Now assume that $y = (f \times \pi_{\mathrm{rat}})(x_1) = (f \times \pi_{\mathrm{rat}})(x_2)$ for some $x_1, x_2 \in W$. Since π_{rat} separates the orbits in X , $x_2 = g(x_1)$ for some $g \in G$. But then $g \in G_y = \{1\}$. We conclude that $x_1 = x_2$. ■

3. Homogeneous Fiber Spaces

Let N be a closed subgroup of a linear algebraic group G , and let W be an N -variety. We first recall the definition of the G -variety $G *_N W$. Consider the action of $G \times N$ on the variety $G \times W$ given by

$$(g, n) \cdot (h, w) = (ghn^{-1}, nw). \quad (1)$$

The variety $G *_N W$ is, by definition, the rational quotient of $G \times W$ for the N -action given by the above formula (where we identify N with the subgroup $\{1\} \times N$ of $G \times N$). Moreover, one can choose a particular model for $G *_N W$ such

that the G -action on $G \times N$ descends to a regular G -action on $G *_N W$ making the rational quotient map $G \times W \dashrightarrow G *_N W$ G -equivariant. The variety $G *_N W$ is often referred to as a *homogeneous fiber space*. (If W is a point with trivial N -action, then $G *_N W$ is the homogeneous space G/N .) Note that above definition of $G *_N W$, taken from [6, 2.12], is somewhat different from the one given in [5, 4.8], where the categorical, rather than the rational quotient is considered. As a result, our $G *_N W$ is only defined up to birational isomorphism (of G -varieties); however, there is no restriction on W (compare with [5, Theorem 4.19]).

For lack of a reference, we include the following easy lemma.

Lemma 3.1. *Let N be a closed subgroup of a linear algebraic group G . If W_1 and W_2 are birationally isomorphic N -varieties, then $G *_N W_1$ and $G *_N W_2$ are birationally isomorphic G -varieties.*

Proof. Say $\varphi: W_1 \dashrightarrow W_2$ is a birational isomorphism of N -varieties. Then $\text{id}_G \times \varphi: G \times W_1 \dashrightarrow G \times W_2$ is a birational isomorphism of $G \times N$ -varieties, so induces a birational isomorphism of the rational quotients by N . One easily checks that this birational isomorphism $G *_N W_1 \dashrightarrow G *_N W_2$ is G -equivariant. ■

We now recall that a subgroup $S \subseteq G$ is called a *stabilizer in general position* for a G -variety X if G_x is conjugate to S for $x \in X$ in general position. Note that if a G -variety X has a stabilizer in general position, it is unique up to conjugacy.

Lemma 3.2. *Let G be a linear algebraic group, and let X be a primitive G -variety. Assume that X has a stabilizer $S \subseteq G$ in general position. Set $N = N_G(S)$, and denote by X^S the set of S -fixed points in X . Let Y be the union of the irreducible components of X^S of maximal dimension. Then X is birationally isomorphic to $G *_N Y$ as a G -variety.*

Our proof is closely related to arguments in [4, Section 1.7]; however, for the sake of completeness, and because we are assuming that X is primitive but not necessarily irreducible, our proof will be self-contained.

Proof. After replacing X by a G -invariant dense open subset, we may assume that G_x is conjugate to S for every $x \in X$. By comparing stabilizers, we see that

$$GX^S = X \tag{2}$$

and

$$\text{if } gx_1 = x_2 \text{ for some } x_1, x_2 \in X^S \text{ and } g \in G, \text{ then } g \in N. \tag{3}$$

We first show that GY is dense in X . Consider the map $\Psi: G \times X^S \rightarrow X$ given by $(g, x) \mapsto gx$. By (2), Ψ is surjective. By (3), the fibers of Ψ are precisely the N -orbits in $G \times X^S$, where N acts by $n \cdot (g, x) = (gn^{-1}, nx)$. Since this action is free (i.e., the stabilizer of every point is trivial), every fiber has the same dimension $\dim(N)$. Since X is G -primitive, the fiber dimension theorem implies that GY is dense in X .

Since Ψ sends N -orbits in $G \times Y$ to points in X , the universal property of rational quotients of N -varieties (see, e.g., [6, Remark 2.4]) says that $\Psi|_{G \times Y}$

descends to a rational map $\psi: G *_N Y \dashrightarrow X$ of G -varieties. Since GY is dense in X , ψ is dominant. We claim that ψ is a birational isomorphism.

Since the irreducible components of Y have the same dimension, and since the dimension of the fibers of Ψ is constant, we conclude that the irreducible components of $G *_N Y$ are also of the same dimension (namely, of dimension, $\dim(G) + \dim(Y) - \dim(N)$). Thus in order to show that ψ is a birational isomorphism, we only need to check that ψ is generically one-to-one. More precisely, we will show that if $\Psi(g_1, y_1) = \Psi(g_2, y_2)$, then (g_1, y_1) and (g_2, y_2) lie in the same N -orbit in $G \times Y$.

Indeed, $\Psi(g_1, y_1) = \Psi(g_2, y_2)$ can, by definition, be rewritten as $y_1 = g_1^{-1}g_2y_2$. By (3), $g_1^{-1}g_2 \in N$. Setting $n = g_1^{-1}g_2$, we see that $(g_1, y_1) = (g_2n^{-1}, ny_2)$, so that (g_1, y_1) and (g_2, y_2) are, indeed, in the same N -orbit. This completes the proof of Lemma 3.2. \blacksquare

4. Proof of Theorem 1.2(ii)

We begin with several preliminary reductions. First note that if $N_G(G_x)$ is reductive then G_x itself must be reductive. Indeed, the unipotent radical $R_u(G_x)$ is trivial, because it is a normal unipotent subgroup of $N_G(G_x)$.

Secondly, we may assume, as we did in the previous section, that X is primitive, i.e., G transitively permutes the irreducible components of X .

Thirdly, by a theorem of Richardson (see [7, Theorem 9.3.1] or [5, Theorem 7.1]), X has a stabilizer $S \subseteq G$ in general position. (If X_0 is an irreducible component of X , and if $G_0 = \{g \in G \mid g(X_0) = X_0\}$, Richardson's theorem immediately implies that the G_0 action on X_0 has a stabilizer in general position; one easily checks that this stabilizer is also a stabilizer in general position for the G -action on the primitive G -variety X .) Now, after replacing X by a G -invariant dense open subset, we may assume that G_x is conjugate to S for every $x \in X$. By assumption, $N = N_G(S)$ is reductive. Denote by X^S the set of S -fixed points in X . Let Y be the union of the irreducible components of X^S of maximal dimension. By Lemma 3.2, X is birationally isomorphic to $G *_N Y$ as G -variety.

Since S acts trivially on Y , we can think of Y as an N/S -variety. By our assumption, G_x is conjugate to S for every $x \in X$. In particular, $G_x = S$ for every $x \in X^S$. Hence, the N/S -action on Y is generically free and, by Theorem 1.2(i), there is a stable affine N/S -variety Z , birationally equivalent to Y . By Lemma 3.1, $X \simeq G *_N Y$ is birationally isomorphic to $G *_N Z$ as G -variety. So it suffices to show that $G *_N Z$ has a stable affine model, namely the categorical quotient $X' = (G \times Z) // N$. Here we identify, as before, N with the subgroup $\{1\} \times N$ of $G \times N$, and the N -action on $G \times Z$ is given by (1).

Since $G \times Z$ is affine and N is reductive, X' is affine. Since the N -action on $G \times Z$ is free, it is stable. Since N is reductive, the categorical quotient map $\pi_{\text{cat}}: G \times Z \rightarrow X'$ separates closed orbits, so all orbits, see, e.g., [5, p. 189, Corollary]. This implies that X' is birationally isomorphic to $G *_N Z$, the rational quotient of $G \times Z$ by N , see, e.g., [6, Remark 2.5].

It remains to show that the G -action on X' is stable. We first note that the $G \times N$ -action on $G \times Z$, given by (1) is stable. Indeed, the $G \times N$ -orbit of $(g, z) \in G \times Z$ is $G \times (Nz)$. Since the N -action on Z is stable, this orbit is closed

for z in general position in Z .

Finally, the G -orbits in X' are images, under π_{cat} , of $G \times N$ -orbits in $G \times Z$. Since π_{cat} maps N -invariant closed sets in $G \times Z$ to closed sets in X' (cf. [5, p. 188, Corollary]), it follows that the G -action on X' is stable. This completes the proof of Theorem 1.2(ii). ■

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