If A.B = 0 then A = 0 or B = 0?

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Dedicamos este trabajo a la memoria de nuestro querido amigo, el Maestro Carlos García

Abstract**:** We present a study carried out in Uruguay, with secondary school students and tertiary level mathematics students, concerning the zero-product property. In our research we observed that when early secondary and late secondary school students have to solve equations of the form $(ax + b)(cx + d) = 0$, they do not always apply the property, even when it is the only available tool and have received specific instruction on its application to the resolution of equations of this type. We also detected an error that students make when they have to verify the solutions of this type of equations. The error consists of the assignment of two different values to the unknown simultaneously. Our study also revealed that late secondary and tertiary level students show a certain tendency to generalize the zero-product property to other algebraic structures where it is not always valid.

Keywords: error analysis; false generalizations; linear equation products; secondary school students; Uruguay; zero-product property

Introduction

According to Bednarz, et al. (1996, p. 3) the introduction of school algebra can take many different directions: “the rules for transforming and solving equations”, “the solving of specific problems or classes of problems”, “the generalization of laws governing numbers”, “introduction

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of the concepts of variable and function" and "the study of algebraic structures". Each one of these options has its conceptual difficulties associated with it and brings about didactical problems to be solved. On the other hand one cannot deny their importance and we may wonder if it is possible to have a curricular approach that reaches a balance among all these possibilities. One way to accomplish this would be to search mathematical topics that can lend themselves to the design of mathematical situations encompassing different facets of algebra. Our research is a step in this direction.

The content of our study touches upon all the approaches mentioned in the above paragraph. On the one hand this makes its mathematical focus a valuable resource for didactical purposes, and on the other hand we get a glimpse into student difficulties concerning the related notions.

Background and research questions

In the title of this article the well-known zero-product property appears in the form of a question. What is the answer to it? And what would happen if we asked it of the students?
Of course the answer depends on the characteristics of the structure to which A and B belong, and it allows us to distinguish those structures that contain divisors of zero from those that do not. This question and the related property have been fundamental in the development of structural algebra.
Texts about the historical development of mathematics (see for example Corry, 1996) show how, little by little, the study of the algebraic structures becomes the main task of Algebra at the beginning of the 20^{th} century. The property on which we focus our attention and the distinction between structures (their classification as having or not having divisors of zero), were particularly important in the development of Abstract Algebra. The zero-product property is the defining characteristic of a type of commutative ring called an integral domain (Wikipedia).

In Uruguay, this property in the context of the real numbers is known as the Hankelian property. Although some textbooks assert that this is due to the name of the mathematician Hankel who discovered it, we could not find any evidence or reference about this claim.
This denomination was, apparently, introduced in Uruguay by a mathematics teacher in a textbook he published in 1958[2]. There the author points out the "brief but deep exposition of Hankel about the theory of numbers an their operations" but he does not make any specific reference that links Hankel with this property[3].

The zero-product property appears along the curriculum in different ways. To illustrate them we will give some examples taking into account the directions stated by Bednarz, et al. (1996, p. 3) that we have mentioned above.
The second degree equation is a common topic of the secondary school curriculum. Usually, teachers present to their students the incomplete forms of the second degree equation before teaching the quadratic formula to solve them. For instance, equations of the form $ax^2 + bx = 0$, can be transformed into $x(ax + b) = 0$. Hankelian property is a useful tool to solve an equation of this type and in general, to solve any second degree equation of the form $(ax + b)(cx + d) = 0$. When students study the quadratic function and they want to find, for example,

the *x*-intercepts (if they exist), the Hankelian property may be a tool to find the roots of the function if its analytic expression is given in an appropriate form.
During early secondary school students study different sets of numbers, the operations defined on each of them and their properties. The Hankelian property is observed in the context of multiplication working with concrete numbers and then it is generalized. Textbooks give students activities such as the following[4]:

Complete:
$\square \times 5 = 0$
$17 \times \square = 0$
$\square \times \square = 0$
$\square \times \square \times 7 \times 3 = 0$
Observe that for the product to be zero at least one of the factors must be zero. In general: If $a \times b = 0 \Rightarrow a = 0$ and/or $b = 0$.

Many times in the late secondary school or at the university, students have the opportunity to study other subjects where they can analyze the validity of the zero-product property in different contexts. For example, in the context of the matrices an Uruguayan textbook –universitary level– gives students the following activity[5]:

Find examples of real 2x2 matrices where:
$B^2 = O, B \neq O$

This activity can lead the students implicitly to realize that the zero-product property is not valid in the context of the matrices.

In our study we wanted to identify student difficulties related to the zero-product property through mathematical situations that address its different aspects. We also wanted to offer didactical strategies that might contribute to a better understanding of this important topic by students at different educational levels.

Three phenomena

In this article we present three didactical phenomena related to this property as we discuss below. First we describe these phenomena shortly and then we elaborate on them, presenting evidence from our research. Since we are exploring different phenomena and what unites them is a mathematical property, we use different viewpoints to provide interpretations for each of them.

<u>First Phenomenon.</u> In our research we observed that when early secondary (14-15 years old) and late secondary school students (17-18 years old) are required to solve equations of the type $(2x - 6)(18 - 2x) = 0$, they do not immediately apply this property, even when it is the only available tool to them and when they have received specific instruction on its application to the resolution of equations where factored polynomial expressions appear equal to zero.

<u>Second Phenomenon.</u> We also detected an error that students make when they have to verify the solutions of an equation such as the one mentioned above. The error consists of the assignment of two different values to the unknown simultaneously (which we will refer to as double assignment), as it is illustrated in the following task carried out by a 17 year-old late secondary level student. The task can be translated as follows:

1) i) Solve the equation $(2x - 6)(18 - 2x) = 0$. Explain how you do it.
   ii) How many solutions did you obtain?_________What are they?______
   iii) Verify the solution(s) that you obtained.

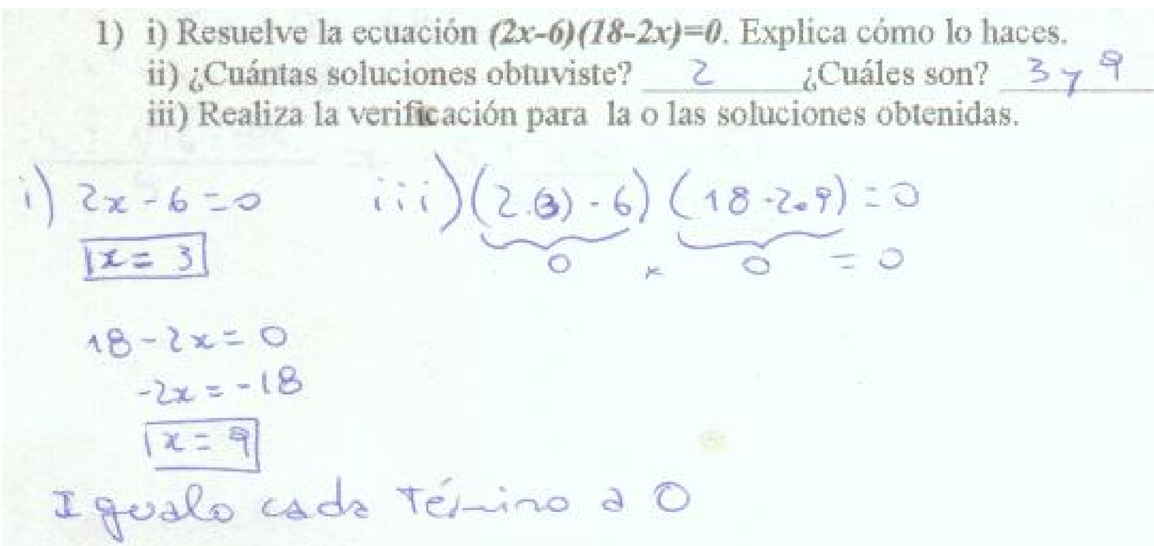


Fig. 1

As can be seen from his work, after finding the solutions as 3 and 9, this student substitutes 3 in the factor $(2x - 6)$, and 9 in the factor $(18 - 2x)$ simultaneously to verify them.

Vaiyavutjamai, et al. (2005) studied the extent to which students from three different nations (coming from Thailand, Brunei Darussalam and the USA) correctly solved equations in the form $x^2 = K$ $(K > 0)$ and $(x - a)(x - b) = 0$ (where a and b can be any real numbers). They report that some students in their research study "tended not to know what their solutions represented in relation to the original equation" (Vaiyavutjamai, et al., 2005). Furthermore, "many students did not realize that if a variable appeared twice in an equation, then it had the same value in the different "places" in which it appeared" (Vaiyavutjamai, et al., 2005). They conclude that the thinking of many students in this task is guided by a misconception related to variables. The authors point out the need for creating a research agenda centered on the topic of quadratic equations as this is an unexplored field in terms of student understanding and difficulties.

A similar phenomenon that corresponds to thinking that the same letter does not necessarily stand for the same value in a given mathematical expression was reported in other studies, as well. Filloy & Rojano (1984) observe that when solving first degree equations such as $x + 5 = x + x$, some students think that the x on the left side of the equation can be any number, but the second x on the right side has to be 5. Fujii (2003) uses expressions such as $x + x + x + x = x$ and $x + x + x = 12$ in a study to illustrate this misconception. In the first case the students are asked whether the expression is correct, and in the second case they are to choose possible correct answers from among three choices provided to them. When students who think that the expression $x + x + x + x = x$ can be correct were questioned about whether "x does not have to be the same number", a student answered by saying "It doesn't have to be the same thing. It's a variable" (Fujii, 2003). The same student who chose (2,5,5) and (10,1,1) as acceptable solutions for the equation $x + x + x = 12$ was questioned whether $x + x + x$ would be replaced by $3x$, and he replied:

> It can, but it can also be wrong. It depends on what x equals, which, because x can equal 10, the first x, and then second x can equal 2. (Fujii, 2003).

According to Fujii (2003, referring to Van Engen, 1961a, b), this misconception stems from the fact that some students consider only the *unspecified* aspect of the concept of variable, and the *definite* aspect, which is in tension with the former, tends to be missing.
In this paper we add another interpretation to this phenomenon, in the context of our research.

<u>Third Phenomenon.</u> We also observed that late secondary as well as tertiary level mathematics students[6] (older than 21 years with various ages) show a certain tendency to extend this property to other algebraic structures where this property is not always valid, as in the context of matrices or real functions.

Erroneous generalization of rules or properties to other contexts where they do not hold true can have its roots in the prior learning experiences of the students with the topic in question, and the intuition that they develop in relation with it (Fischbein, 1987, p.198). For example Aguilar & Oktaç (2004) found that teachers involved in their study tried to solve equations in modular arithmetic structures as if the elements were real numbers and the operations were the usual ones.

Method

In the study that we conducted in order to research the understanding and the use of the zero-product property, we applied a written questionnaire consisting of eighteen questions to two groups (corresponding to 14 early secondary and 14 late secondary level students) and another

one consisting of six questions to two groups of students (corresponding to 10 late secondary and 23 tertiary level students). We interviewed three students from the lower secondary level, seven from the upper secondary level and three from the tertiary level as well as one teacher. The written questionnaires differed slightly depending on the level of the students and the interview probed on those aspects that we considered revealing for the purposes of our research.
The complete questionnaires are given in the appendix. Here we consider a few of the questionnaire items in detail, in order to look into the three phenomena described above. When there are remarkable differences as to the way different groups answer a certain question, we note that, as well.

First Phenomenon. How do students go about solving equations of the form $(ax + b)(cx + d) = 0$?

Kieran (1996, p. 22) distinguishes between three types of activities of school algebra: generational, transformational and global/meta-level. The first one emphasizes the forming of algebraic objects such as expressions and equations, possibly within the frame of a mathematical situation. The transformational type refers to equation solving and manipulation of expressions to get equivalent expressions, among others. The global/meta-level activities are the ones for which algebra is used as a tool such as modeling, noticing patterns and problem solving. As Kieran notes "[a]lgebra textbooks have traditionally emphasized the transformational aspects of algebraic activity, with more attention paid to the rules to be followed in manipulating symbolic expressions and equations than to conceptual notions that support these rules or to the structural underpinnings of the expressions or equations being manipulated" (Kieran, 1996, p. 24).

In our research we observed that when early secondary students try to solve $(ax + b)(cx + d) = 0$ type equations, they usually apply the distributive law and/or try to use some well-known technique for solving first degree equations, as they do not yet know the quadratic formula. The procedure they apply is usually erroneous; below we present examples of students' work illustrating some of these strategies:

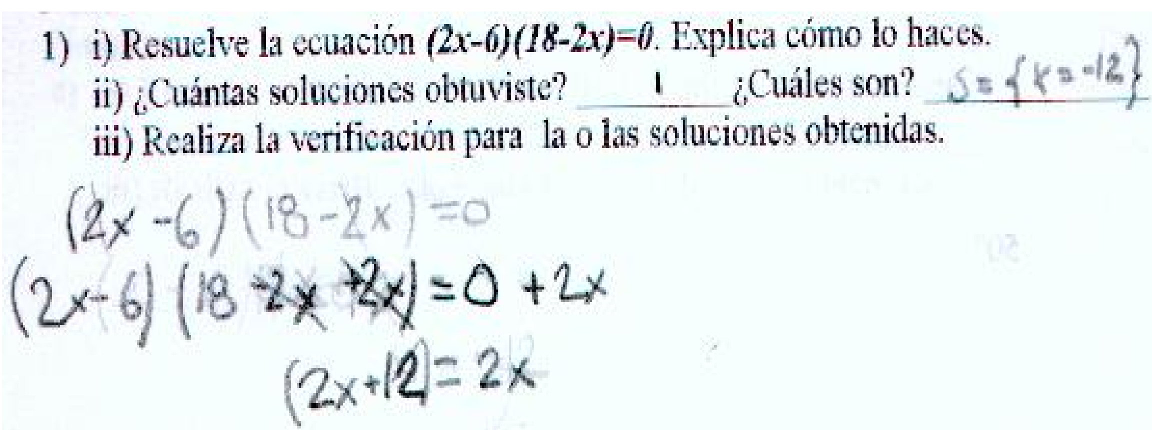
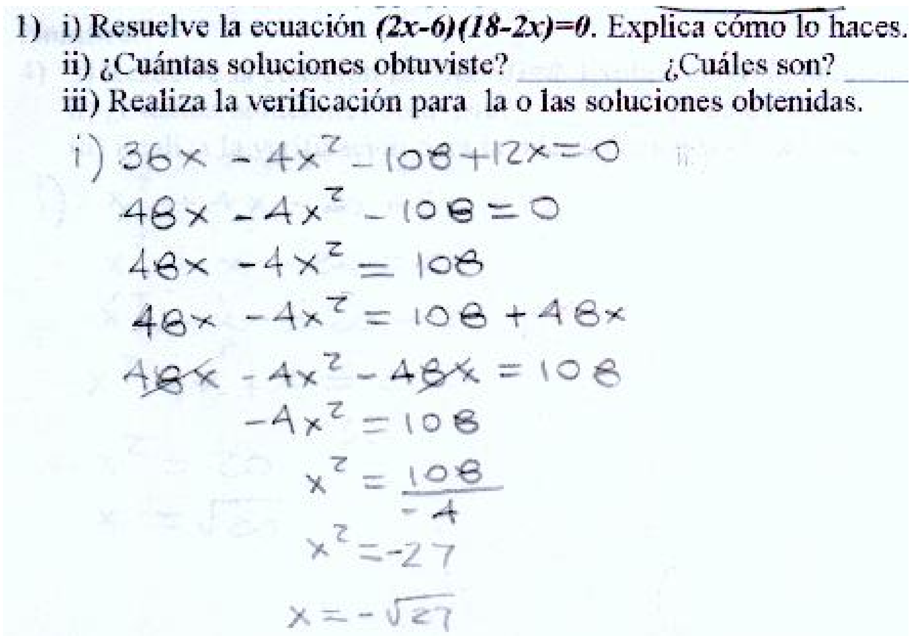


Fig. 2

In Fig. 2 the student introduces 2*x* on both sides of the equation, however on the left side it is added within one of the factors of the product.

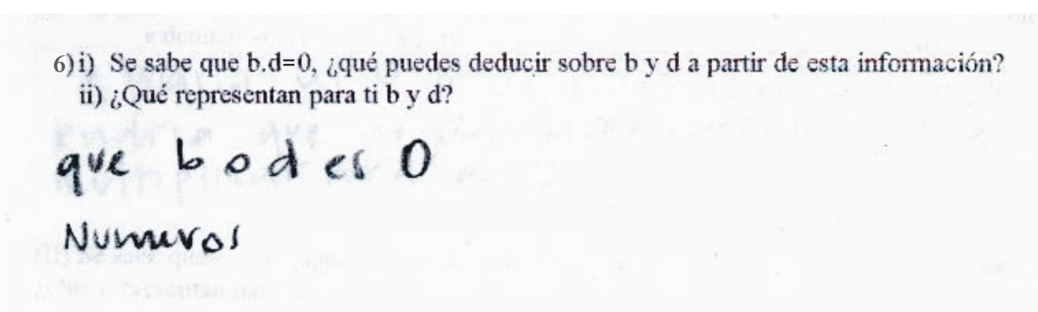

1) i) Resuelve la ecuación ***(2x-6)(18-2x)=0***. Explica cómo lo haces.
ii) ¿Cuántas soluciones obtuviste? ______ ¿Cuáles son? ____
iii) Realiza la verificación para la o las soluciones obtenidas.

i) $36x - 4x^2 - 108 + 12x = 0$
$48x - 4x^2 - 108 = 0$
$48x - 4x^2 = 108$
$48x - 4x^2 = 108 + 48x$
$48x - 4x^2 - 48x = 108$
$-4x^2 = 108$
$x^2 = \frac{108}{-4}$
$x^2 = -27$
$x = -\sqrt{27}$

Fig. 3

Fig. 3 shows the work of a student who introduces the term 48*x* to the right side of the equation, without doing the same thing on the left side. This allows him to get rid of the term involving *x* and making it possible to arrive at an "answer".

About five months before they completed this questionnaire, these students had been instructed to solve this type of equations by applying the zero-product property. However, they do not seem to recognize its applicability even though they apparently know about it, as we see in the following type of answers:


6) i) Se sabe que b.d=0, ¿qué puedes deducir sobre b y d a partir de esta información?
ii) ¿Qué representan para ti b y d?

que b o d es 0

Numeros

Fig. 4

The question in Fig. 4 reads:

6) i) We know that $b.d = 0$. From this information, what can you conclude about b and d?

ii) What do b and d represent for you?

For (i), the student writes: “that b or d is 0”. For (ii), her answer is: “numbers”.

Most of the students are not successful in the task of solving $(2x – 6)(18 – 2x) = 0$ due to the complexity of the equation that they obtain after applying the distributive law, as they do not have resources like the quadratic formula to solve it (they have not studied it yet). As an example, in the following figure we can see the work of a student whom we will call Clarise, an early secondary student:

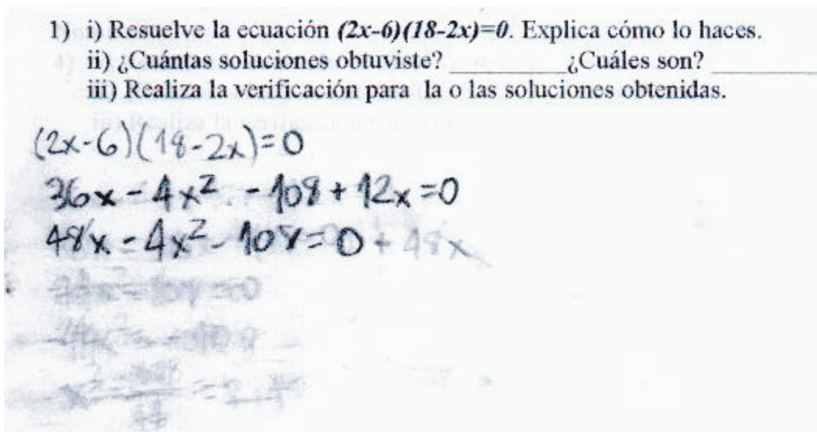


Fig. 5

Although in some cases students said that they could give the solutions of the equation mentally without carrying out any operation, it seems that this does not satisfy them because they have not applied an algorithmic procedure to solve it. We can see this in the following translation of Clarise´s interview, which revealed that actually she knew the answer:

[…]

Interviewer: So, you, according to what you answered in question number 6 (see Fig. 4), you knew that if the product of two factors is zero, then one or the other must be zero.

Clarise: Yes.

I: Nevertheless you didn’t apply it to solve the equation (see Fig. 5). Can you explain why?

C: Because then I started thinking about it, right? It was something that I was deducing without performing any operation. Let’s say I was able to find it, what I couldn’t do was by seeing, I mean, solving for the *x*.

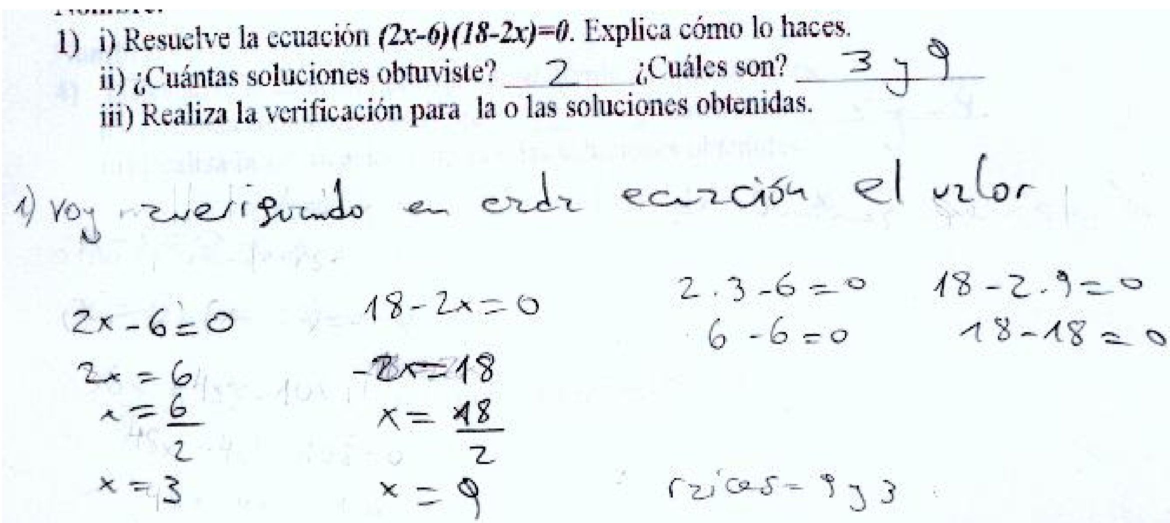
I: So, the idea that you just explained to me, that *x* must be 3 or 9, you knew it well...

C: Of course, what I didn’t know was how to arrive at the result using a formula, how can I tell you? I couldn’t follow an operation, I couldn’t find according to all the procedures that *x* was equal to 3 or *x* was equal to 9.

In some cases this could be due to the fact that mathematics is taught as a set of rules or procedures to apply, giving more emphasis to the transformational type of activities (Kieran, 1996, p. 24). When students face a situation that is not familiar to them, they will try to apply certain algorithms known to them, since they believe that this way they will surely reach a solution – as they do, on occasions.

When Lima and Tall (2006) asked 15-16 year old students to solve equations of the type $(y - 2)(y - 3) = 0$ among others, not only no one mentioned or used the zero-product rule, but "students did not seem to believe it. The only met-befores seem to be numeric 'guess and test' to seek solutions, or an attempt to use the quadratic formula. The students therefore are at a procedural level relying on a single procedure, without the appreciation of several procedures to give alternative approaches" (Lima & Tall, 2006).

In our study only a small percentage of students was able to solve the equation in question, equating each factor to zero in order to find the roots. These students could also formulate an explanation for what they were doing as we see in the work shown in Fig. 6:


1) i) Resuelve la ecuación *(2x-6)(18-2x)=0*. Explica cómo lo haces.
ii) ¿Cuántas soluciones obtuviste? 2 ¿Cuáles son? 3 y 9
iii) Realiza la verificación para la o las soluciones obtenidas.

1) voy averiguando en cada ecuación el valor

$2x - 6 = 0$, $2x = 6$, $x = \frac{6}{2}$, $x = 3$

$18 - 2x = 0$, $-2x = 18$, $x = \frac{18}{2}$, $x = 9$

$2 \cdot 3 - 6 = 0$, $6 - 6 = 0$

$18 - 2 \cdot 9 = 0$, $18 - 18 = 0$

raíces = 9 y 3

Fig. 6

Some late secondary students that apply this property to solve the equation do so because their teachers have taught them that "*it is done this way*" or because they were *"taught to do it this way*" (as they explained themselves) and they cannot always offer an explanation with mathematical arguments for what they do. But in general, the strategy preferred by students at this level consists of developing the polynomial expression and applying the quadratic formula as we can see in the following figure. Even when the student knows the two procedures (application of the property and the use of the quadratic formula), he/she does not choose the application of the property as a simpler procedure.

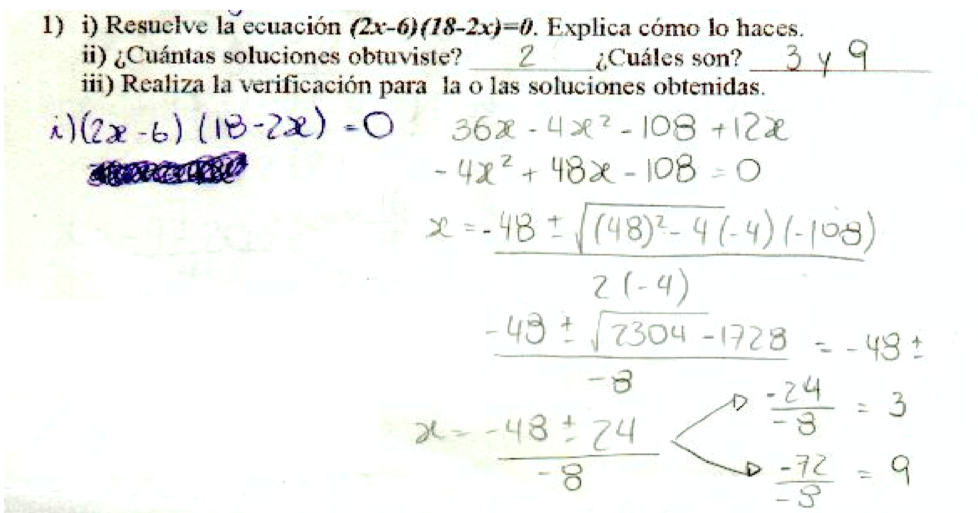


Fig.7

On the other hand the tertiary level mathematics students have generated enough autonomy to decide and to choose what tool to use according to the situation they face. In our study they always applied the property whenever the equation allowed it.

Second Phenomenon. Verification of the solutions of an equation of the form $(ax + b)(cx + d) = 0$

Although verification and validation of solutions is an important part of the problem solving process, students usually don't feel the need to check their answers. Furthermore, even though in a problem-solving context validating an answer might make sense to the students, in an algebraic setting there is little meaning given to this type of activity.

In Uruguay when early secondary students begin to study first degree equations, teachers usually use the verification as a way to explain them what it means for a specific number to be a solution of an equation. Students' textbooks also use the verification in the same sense.
In the case of the verification of the solutions of an equation of the form $(ax + b)(cx + d) = 0$, textbooks give students exercises such as the following[7]:

> ¿True or false?
>
> In the equation $(x - 1)(x + 2) = 0$ the numbers -2 y 1 are roots.

Nevertheless, none of the students' textbooks show how to do the verification of an equation of this form. In the classroom, it very much depends on the course instructor whether students would engage in an activity of verification.

When asked, students frequently verify the solutions of this type of equations replacing the *x* of the first factor by – *b/a* and that of the second factor by – *d/c* simultaneously. We detected this strategy at all educational levels at which we applied our instrument, that is at early and late secondary, and tertiary levels. A priori we had supposed that it was closely related to the application of the property: We think that the student could believe that the second degree equation is fragmented into two first degree equations and therefore he/she does the verification this way. In the interviews the students manifested certain confusion in relation to whether the equation $(2x – 6)(18 – 2x) = 0$ was one equation or two equations. As the student ends up solving two first degree equations, it is possible that he/she treats the whole equation as two linear equations even when the substitution has been made in an equation of second degree, without the student being aware of it. In Fig. 8 we see an example illustrating this phenomenon.

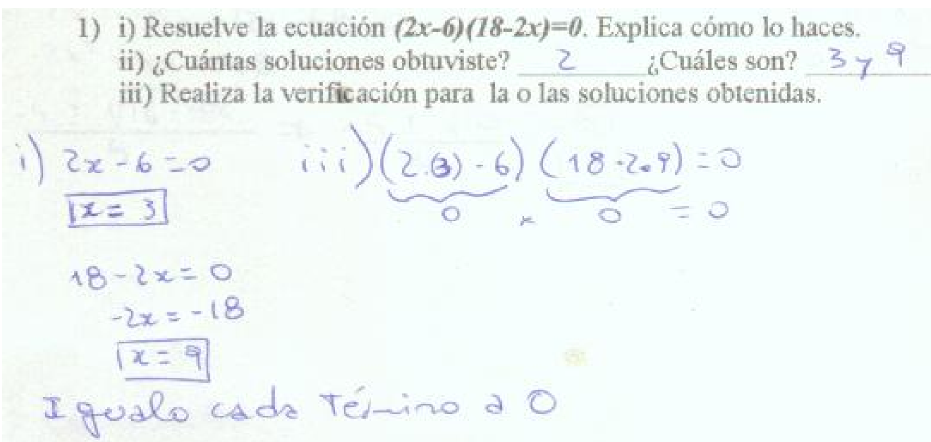


Fig. 8

We saw that this error is also present when other strategies such as the application of the quadratic formula are used to solve the equation. We can see this in the following work of a late secondary level student:

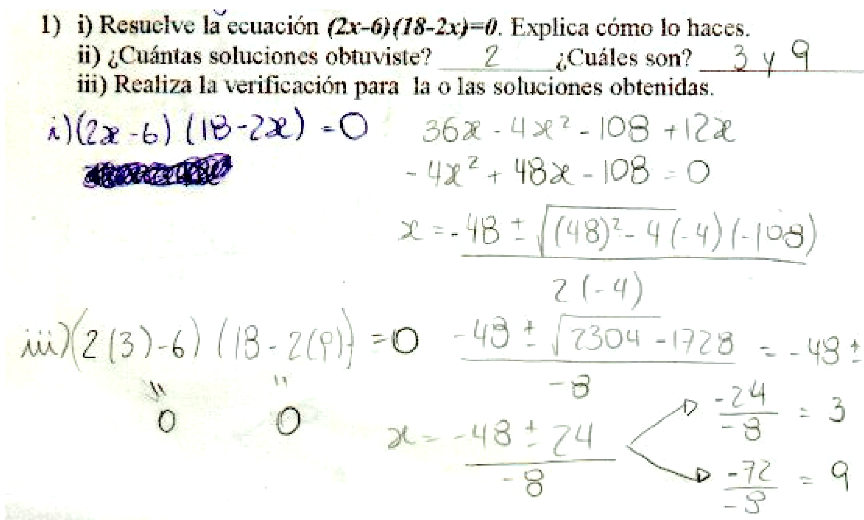


Fig. 9

One might think that the students may be using this strategy as a shortcut, however we observe that although the students claim that if a product is zero then one of the factors should be zero, a strong belief seems to exist that both factors should be simultaneously zero. This situation that we consider of an intuitive nature might condition how the student thinks that the concept of variable functions, leading him/her to make the double assignment. This belief was evidenced through the following activity where most of the early and late secondary level students replied that the numbers were 6 and 19. The activity reads:

13) The papers are hiding numbers, can you find them? Explain your reasoning.

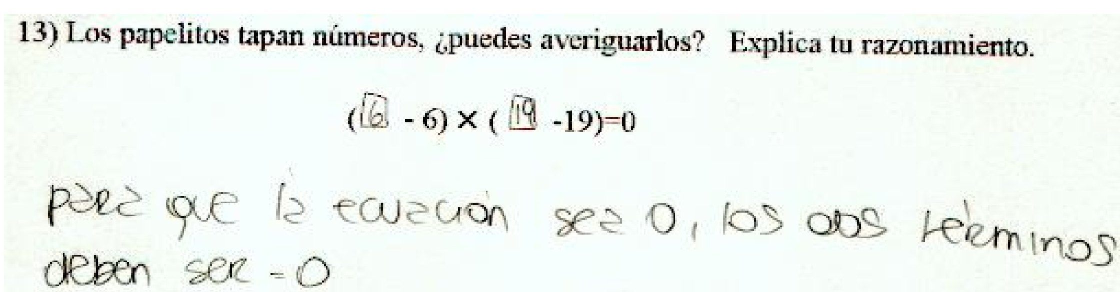


Fig.10

This student wrote: “In order for the equation to be 0, both terms should be = 0”.

The error was not exclusive to the equation that is given in factored form but it also appeared when verifying the roots of an equation whose polynomial expression was developed. The following activity can be translated as:

> 17) Are 3 and 4 roots of the equation $x^2 - 7x + 12 = 0$?
> Explain your answer, clarifying whatever you think is appropriate.

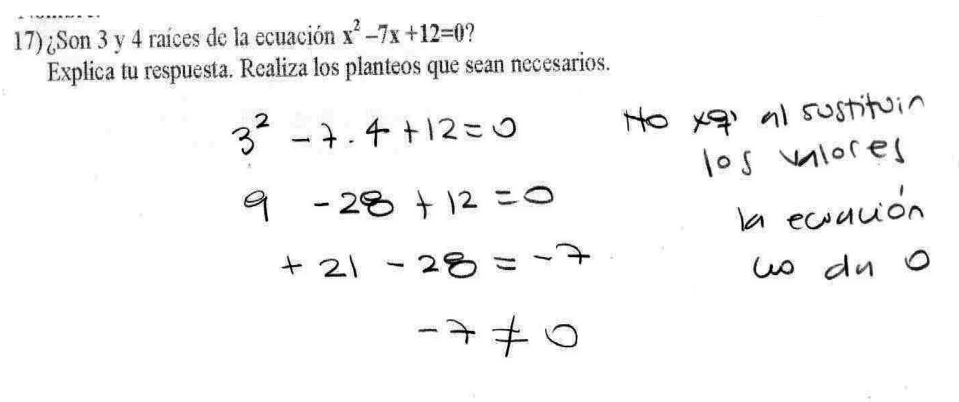


Fig. 11

The student's answer is: "No, when the values are substituted the equation doesn't give 0". We observe that here 3 was substituted in the x^2 term of the equation, and 4 was substituted in the $-7x$ term. It seems that this student wants to make sure that 3 **and** 4, being two roots of the given equation, appear in the verifying process simultaneously. This might cause a conflict with the solving process when we get "$x = 4$ **or** $x = 3$" as a result.

If the students find the roots of the equation themselves using a procedure and if they use the developed expression to verify them by substituting two different values simultaneously, they are more likely to detect that there is something that is not working since they do not obtain an expression of the type $0 = 0$. On the other hand if they use the factored expression they will obtain $0.0 = 0$, in which case they are not likely to realize the error.

Vaiyavutjamai and Clements (2006) report about Thai students who thought that the two x's in the equation $(x - 3)(x - 5) = 0$ represented different numbers (writing $(3 - 3)(5 - 5) = 0$ to check their answers) and even after an instructional treatment continued with this belief. They observed the same type of phenomenon when they used the equation $x^2 - x = 12$. In this case students were wondering why they were not getting an equality when they substituted different numbers for the two x's.

In order to observe if the students would detect their error, we prepared a contextualized situation where it was impossible that the variable (in this case the hour of the day) could take two different values at the same time. For that, we worked in the context of real functions and the calculation of their images.

To evaluate the need for traffic lights at an intersection, a device that counts cars is placed. The results of the study are given by the expression:

$$A = (2x - 6)(18 - 2x)$$

where A represents the number of cars that go through the intersection at the hour x.

a) How many cars crossed the intersection at 5 o'clock? Show your calculations.
b) Calculate how many cars crossed at 3 o'clock and at 9 o´clock. Show your calculations.
c) At what time did no car cross? Why?

Let's see an extract from Martín's interview, a late secondary student. After Martín claims that to verify if 3 and 9 are roots of the equation $(2x - 6)(18 - 2x) = 0$ he should substitute 3 in the first factor and 9 in the second, he was presented with the traffic lights problem. After he solves this problem correctly, the interviewer probes more:
[...]
Interviewer: If I ask you how many cars cross the intersection at 3 in the morning and at 9 in the morning, what would you do to solve this Martin?
Martín: Substitute both *x* by the same number, that is, by the same root, both by 3 or both by 9.
T: Right. So, between this attempt and this one[8], which one you consider appropriate to answer the problem of the cars and traffic lights?
M: (Points out the correct one)
I: And now if we leave the context of this problem and forget that we are talking about cars and traffic lights and we just want to know if 3 and 9 are roots of the equation…
M: Yes.
I: There, you can do...
M: I can do any of the procedures.
I: And in the context of the problem, why did you think that *x* could not value at the same time 3 in the morning and 9 in the morning?
M: In the context of the cars?
I: Yes.
M: Because we are talking about hours. It can' t be, It wouldn' t be a possible situation.

The student pointed out that in the case of this problem *x* should take the same value because it could not be 3 in the morning and 9 in the morning at the same time. However in verifying the roots of an equation it could, since according to them *x* was not representing the hour of the day in this case. In this way the student isolated the case of the verification of the roots of the equation as a special situation, separating it from the calculation of image values of a function. In this way he was maintaining the coherence of their mental schemes. We could say that in this case, the compartmentalization phenomenon (Vinner, 1990) appears as a resource of the mind that allows the student to avoid contradictions and therefore to maintain internal coherence, isolating the two situations and recognizing them as different things that are not connected.

The evidence in our work shows that the verification process is not obvious, and neither is the idea that the unknown cannot be replaced at the same time by two different values.

The assignment of different values to the same variable was a resource used by the youngest students when they were asked to build an equation with two given roots. We see it in the following task that reads:

11) Give an equation with 4 and 3 as its roots. How do you do it?

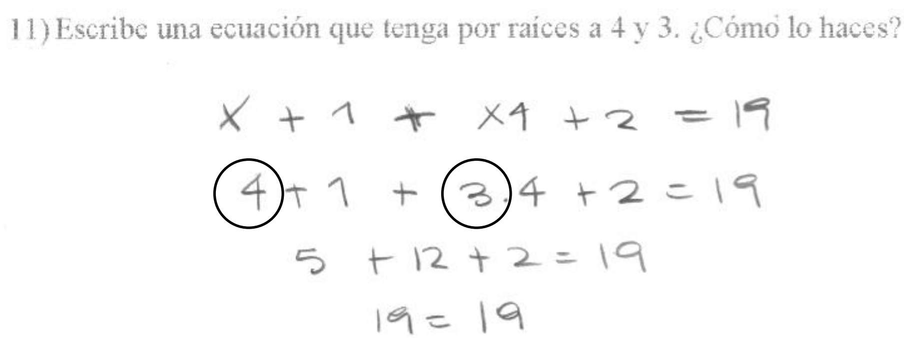


Fig.12

This procedure and the one that considers an equation in two variables, such as $(x - 4)(y - 3) = 0$, assigning to x the value 3 and to y the value 4, was more natural or more spontaneous for them than conceiving a second degree equation in one variable that had the two given roots. Only three students out of fourteen wrote a second degree equation with one variable to answer this task. Maybe conceiving a second degree equation with one variable with the possibility of the existence of two roots is much more complex for the students than educators might believe. With reference to this, Trigueros and Ursini (2003) point out that the great majority of the students with whom they worked thought that the unknown involved in a quadratic equation could take only one value.

Third Phenomenon. Finally… does A.B = 0 imply that A = 0 or B = 0?

Another aspect on which we centered our attention was whether the students extended the zero-product property to structures where this is not valid. We also wanted to shed light on the reasons for doing so, even in the case when they had received specific instruction on this subject.

To go about finding this out, we asked the students, among others, three questions that we consider key:

- ✓ *It is known that b. c = 0. Based on this information, what can you conclude about b and c?*
  What do b and c represent for you?
- ✓ *f and g are two functions whose domain is R. It is known that f.g = O, that is to say that the product of f and g is the zero function. Based on this information, what can you conclude about f and g?*
- ✓ *D and B are two matrices. It is known that D.B = 0, that is to say that the product of the two matrices is the zero matrix. Based on this information, what can you conclude about D and B?*

The first question refers to two factors *a* and *b* whose product is zero, where the nature of *a* and *b* is not specified. The second question is about two functions whose product is the zero function, and the third one involves two matrices whose product is the zero matrix. In each case the students were asked what they could conclude about the factors.

When they took the questionnaire, the late secondary students in our study had already taken a course in Analysis and another in Algebra and Analytic Geometry. The tertiary level students had also studied those subjects at the university level. In the case of the functions, we do not know whether the students had seen examples of non-zero functions whose product was the zero function. On the other hand, in the case of matrices all students who took the test had seen that the property was not valid. In the three cases that we present here, a high percentage of the students answered that one of the factors was zero in the first question, was the zero function in the second question and the zero matrix in the third question.

Several late secondary school students reached their conclusions thinking about the property of absorption, that is, if one of the two factors is zero, then the product will also be zero. They did not realize that the inverse property does not necessarily hold in all the structures.

In the case of the tertiary level students, the validity of the zero-product property in the context of real numbers greatly influenced their answers. In the interviews they stated explicitly that when they gave their answer (that one of the factors must be zero) they thought of this property in the set of the real numbers. For these students this property appears extremely linked to experience, in the sense that they try to apply it whenever possible, factoring in a convenient way, as they themselves pointed out.
Besides the experience in applying the property in familiar situations and contexts, we consider that its incorrect extension is favored by the textbooks that give the rule as "if $a.b = 0$ then $a = 0$ or $b = 0$", without specifying what *a* and *b* represent, and without warning that its validity is not

universal. For example, in an Uruguayan textbook[9] for early secondary students we find the following statement about the multiplicative property of zero:

$a \times 0 = 0 \times a = a \quad \forall a \in N$	PROPIEDAD DE ABSORCIÓN

Later in the same page, the following statement appears:

Si $a \times b = 0 \Rightarrow a = 0$ *y/o* $b = 0$	PROPIEDAD HANKELIANA

We can see that in the case of the multiplicative property of zero (known in Uruguay as the property of absorption) what "*a*" represents is clearly specified (in this case a natural number). However the same thing does not happen in the zero-product property that is frequently known in Uruguay, as we have already said, as the Hankelian property.
This could lead the students to consolidate a thought model that does not include the nature of the objects *a* and *b*, fixing the attention only in the syntax of the writing. According to English and Halford (1995, p. 230), students frequently generate 'malrules' by constructing prototype rules whose surface structure corresponds to the writing of a property. We think that the syntactic features of the writing would favor the application of the property by students, in contexts where it is not valid. While the mathematical objects change, the visual syntactic features remain practically unchanged, giving rise to a mental image such as:
@ . € = 0 ⇒ @ = 0 or € = 0, where the symbols @ and € can be replaced with anything.

We think that the rule in question could have the characteristics of an implicit model of thought which is based on the visual syntactic features of the expression involved. Since in their experience students reinforce constantly the validity of that rule in the context of real numbers, they find it difficult to incorporate further and new information that goes contrary to this experience (Fischbein, 1987, p. 194-195), as in the case of the multiplication of matrices.

Another possible interpretation can be made if we refer to the work of Tirosh and Stavy (1999) about intuitive rules: Students could be applying a rule of the type "Same hypothesis (null product of two factors) - Same conclusion (one of the factors is null)", without paying attention to the semantic aspects of the mathematical objects involved. In particular, the generalization of this property to the case of the first question above, where the nature of the factors was not specified, seems to support this hypothesis.

Later, in order to observe the reactions of early and late secondary level students before a structure that admits divisors of zero, they were given a sequence of activities on residual classes modulo 6. None of the two groups had previous experience with this topic. Here we present part of the sequence:

***(I)** We will work with the elements of set A. The set A is the following:*

$$A=\{0, 1, 2, 3, 4, 5\}$$

We will define an operation that we will call multiplication and we will represent it with the symbol: •

This operation works according to the following table:

•	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

For example, to compute 2 • 5 *you have to look for the intersection of the line and the column as shown in the table above, and you obtain* 4 *as a result.*

a) Using the multiplication table that appears above, calculate:

4 • 2 =

3 • 3 =

5 • 4 =

b) Using the table, find the values of x that satisfy the equation 3 • *x* = 0

Write here the value or the values you have found for x ....

Explain what you did.

c) a and b are elements of the set A as above. It is known that a • b = 0; based on this information, what can you say about a and b? Explain your answer.

In this activity students were asked what they could conclude about two factors whose product is zero ((Ic) of the sequence). While the youngest students took into account the existence of divisors of zero giving correct responses, most of the late secondary students gave the answer that one of the factors had to be zero. Hence the youngest students demonstrated a more versatile thought than that of late secondary level students; this might be because the experience with structures without divisors of zero had still not greatly influenced the consolidation of stable schemes of thought.

From the point of view of the didactical strategies that are commonly used in classrooms, the arithmetic operations and their properties constitute the entrance to the understanding of the algebraic operations and their properties. From a cognitive point of view this first interpretation could block later and more abstract generalizations as Fischbein (1987, p.198) points out. Let us add to this the epistemological element that we find in the work of the mathematician Peacock (1791-1858) when he built the axioms of the symbolic algebra starting from the fundamental laws of arithmetic. According to Boyer (1992, p. 711), Peacock provoked, unintentionally, a stagnation in the evolution of algebra when he institutionalized the universal validity of these laws because he suggested that they remain the same and do not depend on the mathematical object. The laws he wrote did not include, for example, the existence of operations that were not commutative.

Didactical suggestions

In relation with the first phenomenon (not applying the zero-product property) and on the evidences obtained in our study, we can suggest that educators make more emphasis on how an equation is solved rather than on why it can be solved more efficiently in one way or another. As a pedagogical strategy, we think that before early secondary students are taught to apply the zero-product property to solve equations, they can be faced explicitly with appropriate second degree equations, given both in expanded and factored forms. In this way, they can realize that the *sui generis* procedures that they use to discover the unknown, are in most of the cases ineffective. This way, on the one hand they may value the tool that is to be taught, and on the other hand they can see that it is not always necessary to carry out a long sequence of operations to be able to solve an equation, which is what many of them believe. Of course this strategy has to be coupled with methods to allow the students to see whether a particular answer is correct or not.

About the second phenomenon (making a double assignment in verifying) we can say that the students who participated in this study seemed to have it clear that the variable can take one value at a time when calculating the image values of a function. We thus suggest that a possible didactic alternative to avoid this error could be to teach the resolution of equations in the context of functions; that is, ask the students to find the roots of a real function f given in the form $f(x) = (ax + b)(cx + d)$. This way the students could solve the equation $(ax + b)(cx + d) = 0$ in order to find the roots of the function f. To verify that the real numbers they have found are, in fact, the roots of the function the students could analyze if each one has image zero under f. In this way they would be calculating images, so we think that perhaps the double assignment would not occur since they would be focusing on a single root at a time. However more research is needed to find out if this approach would result in a different outcome.

As a didactic suggestion in relation with the third phenomenon (generalizing the zero-product property to other structures) we propose the possibility that the study of Algebra not only begins starting from the arithmetic operations and their properties, but also puts the students in contact with other structures that are within their reach and that offer them a wider vision of the algebraic properties. For example, after students work with natural numbers and whole number division, we can introduce the concept of residual classes to early secondary students, by means of activities such the following (Ochoviet, 1999):

The company that supplies mineral water "Coolish", divided the city of Montevideo in 164 zones in order to make the punctual weekly delivery of this bottled water in each home.
Mrs. Mary Jo, who organizes the delivery, established the following chronogram taking into account the number of the zone.

Monday	*Tuesday*	*Wednesday*	*Thursday*	*Friday*
1		*2*		*3*
	5		*4*	
6		*7*		*8*
	10		*9*	
11		*12*		*13*
	......		......	
......		......		......

Find out which day of the week the company delivers the mineral water to zone 164.

After students work with activities that lead them to the concept of residual classes we can offer them others such as the one we used in the present research, where students worked with residual classes modulo 6 and experienced the existence of zero divisors.
This way we believe that the obstacles can be minimized so that in future the students can conceptualize more abstract or general structures.

Final comments

Understanding of the zero-product property which has been the subject of this article is very important for students at all levels, starting with the secondary level. Furthermore it is a topic that encompasses different facets of algebraic activity and can serve as a source for design of suitable mathematical situations for different age levels. Difficulties associated with it can be related to the understanding of the concept of variable, structural thinking in Algebra and the understanding of procedures that are linked to mathematical properties. Further research can point out the nature of these relationships and focus on the design of appropriate didactical strategies to overcome possible obstacles.

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Endnotes

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[7] Belcredi, L. y Zambra, M. (1998). *Gauss 3*. Montevideo: La Flor del Itapebí. Pp. 38.
[8] The teacher refers to the correct attempt (one assignment at a time) and the wrong one (double assignment), as both have been done by the student at different moments of the interview.
[9] Borbonet, M., Burgos, B., Martínez, A. & Ravaioli, N. (1995). *Matemática 1*. Montevideo: Editorial Fin de Siglo. Pp.23. See also Apostol, T. (1972). *Calculus*. Volumen 1. (2ª Ed). España: Editorial Reverté S. A. Pp. 23.

APPENDIX

Questionnaire 1

1) i) Solve the equation $(2x - 6)(18 - 2x) = 0$. Explain how you do it.
ii) How many solutions did you obtain?_________What are they?______
iii) Verify the solution(s) that you obtained.
2) i) Solve the equation $(x + 6)(2x - 8) = 0$. Explain how you do it.
ii) How many solutions did you obtain?_________What are they?______
iii) Verify the solution(s) that you obtained.
3) i) Solve the equation $(3x - 6)(x - 7) = 0$. Explain how you do it.
ii) How many solutions did you obtain?_________What are they?______
iii) Verify the solution(s) that you obtained.
4) i) Solve the equation $(x - 5)(x + 4) = 0$. Explain how you do it.
ii) How many solutions did you obtain?_________What are they?______
iii) Verify the solution(s) that you obtained.
5) i) Solve the equation $(x - 9)(x - 6) = 0$. Explain how you do it.
ii) How many solutions did you obtain?_________What are they?______
iii) Verify the solution(s) that you obtained.
6) i) Solve the equation $x(2x - 10) = 0$. Explain how you do it.
ii) How many solutions did you obtain?_________What are they?______
iii) Verify the solution(s) that you obtained.
7) i) Solve the equation $x(x - 8) = 0$. Explain how you do it.
ii) How many solutions did you obtain?_________What are they?______
iii) Verify the solution(s) that you obtained.
8) i) Solve the equation $x^2 = 6x$. Explain how you do it.
ii) How many solutions did you obtain?_________What are they?______
iii) Verify the solution(s) that you obtained.
9) i) Solve the equation $5x = 0$. Explain how you do it.
ii) How many solutions did you obtain?_________What are they?______
iii) Verify the solution(s) that you obtained.
10) Give an equation with 8 as a root.
11) Give an equation with 4 and 3 as its roots. How do you do it?
12) In the equation $(2x - 4)(......) = 0$ we do not know the second factor.
Is 2 a root of the equation? Why?
Is 3 a root of the equation? Why?
13) The papers are hiding numbers, can you find them? Explain your reasoning.

$$(\square - 6)(\square - 19) = 0$$

14) Is 7 a root of the equation $(3x - 21)(x - 3) = 0$? Explain your answer.
15) Are 6 and 2 roots of the equation $(2x - 12)(5x - 10) = 0$? Explain your answer.
16) Are 5 and 4 roots of the equation $(2x - 10)(3x - 8) = 0$? Explain your answer.
17) Are 3 and 4 roots of the equation $x^2 - 7x + 12 = 0$?
Explain your answer, clarifying whatever you think is appropriate.
18) i) We know that $b.d = 0$. From this information, what can you conclude about b and d?
ii) What do b and d represent for you?

Questionnaire 2

1) Solve in R the equation $(2x - 6)(5x + 10) = 0$.
Verify the solution(s) that you obtained.
2) It is known that $b.c = 0$. Based on this information, what can you conclude about *b* and *c*? What do *b* and *c* represent for you?
3) *f* and *g* are two functions whose domain is R. It is known that $f.g = O$, that is to say that the product of *f* and *g* is the zero function. Based on this information, what can you conclude about *f* and *g*?
4) *D* and *B* are two matrices. It is known that $D.B = 0$, that is to say that the product of the two matrices is the zero matrix. Based on this information, what can you conclude about *D* and *B*?
5) *p* and *q* are two polynomials. It is known that $p.q = o$, that is to say that the product of the two polynomials is the null polynomial. Based on this information, what can you conclude about *p* and *q*?
6) It is known that $b.c = 0$. Based on this information, what can you conclude about *b* and *c*? What do *b* and *c* represent for you?