

Teaching Proof at KS4

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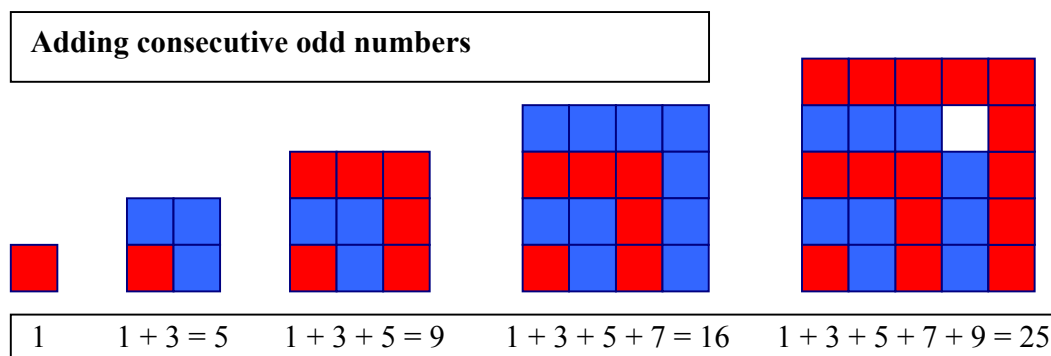
Proof is a fundamental component of mathematics and so, in my opinion, it should be part of mathematical education in schools. It is an important link between topics and so can help pupils achieve a deeper understanding of the “wholeness” of mathematics and, moreover, it is vital for able pupils if they are to be inspired by mathematics and given a firm basis for future mathematical studies.

Ideally, proof in primary schools would take the form of explanations of (mainly) number properties and patterns and the language used could be diagrams, or even pictures. Even in secondary schools much proof would be informal but older, able pupils should be exposed to formal proofs, probably including some from Euclid, and also be made aware of different proof methods. Because this ideal has not always been achieved I have found it necessary to introduce some older pupils to ideas about proof during the latter part of their schooling.

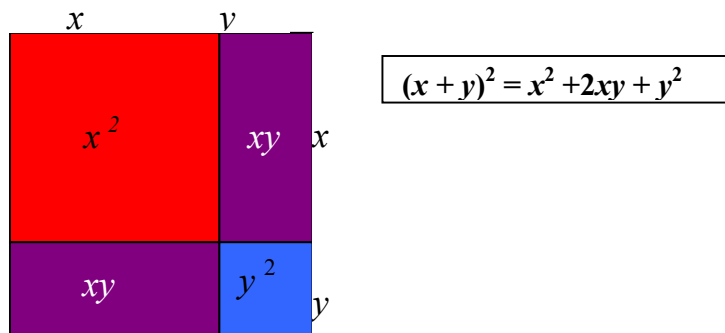
I have provided proofs, mainly deductive, for nearly all the traditional mathematics they have learned and I have also explained why they cannot yet prove the few exceptions, like the formula for the volume of a sphere (*they do not study calculus*). To cater for the needs of pupils who prefer a visual analysis I have used proofs based only on diagrams as proposed by Skemp (1971, p 99), alongside the conventional verbal-algebraic forms, of some geometric theorems. Diagrams have also been used to explain some numerical and algebraic relationships and some of these are shown below.

Diagrams as proof

Pupils who fail to recognise a sequence of square numbers frequently resort to “adding the next odd number” and the first diagram explains why this works.



Many pupils learn algebraic identities by rote and often use them with little real understanding and so the next diagram can help to increase insight.



Proof and pattern

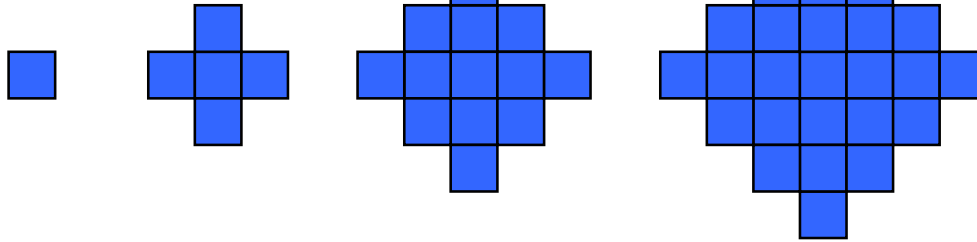
Although traditional proofs and “proofs without words” can deepen understanding of the mathematical concepts they explain they do not necessarily enhance appreciation of the need for and the nature of proof. Even some able pupils do not have an intuitive grasp of this. Using the context of pattern separates proof from the constraints of formal mathematics, such as symbolic language and unknown varying lengths and angles, and so makes it more accessible to more pupils. Patterns are either visual or composed of discrete quantities, which can be represented by concrete apparatus or diagrammatically, and explaining them does not necessarily involve formal mathematical language.

Pupils can often analyse a pattern and work out the next three terms and most able pupils can also work out a general rule. In order to do this they assume that the numerical relationships observed in the first few terms continue. However, even amongst able pupils, there are many who are taken by surprise when asked to explain why the rule works. To alert pupils to the fact that not all patterns continue as expected and to emphasise that the purpose of proof is to provide confidence in the truth of a claim it is wise to use at least one example where an apparent pattern does not continue.

The investigation “Regions in a Circle”, described in my book “Can you prove it?” (Waring, *published by the Mathematical Association, 2000, p157*), uses a counter-example to demonstrate this. Pupils count the number of regions (r) formed by joining with straight lines all possible pairs of dots around the circumference of a circle. Most pupils describe the pattern observed in the first five circles as “doubling” and some can also express it algebraically as $r = 2^{d-1}$. Pupils expect that there will be 32 regions for six dots and are surprised that 31 is correct and that the apparent pattern fails. Pupils are not expected to find the correct complex pattern, but should be aware that one exists and offered copies of a proof (e.g. Beavers, 1994, p10) to read

There are many other examples of patterns to explain in my book and they are accompanied by details of how they have been used in the classroom. One example (part of a pupil worksheet), which has the advantage of having several alternative proofs, is given below.

Growing Squares

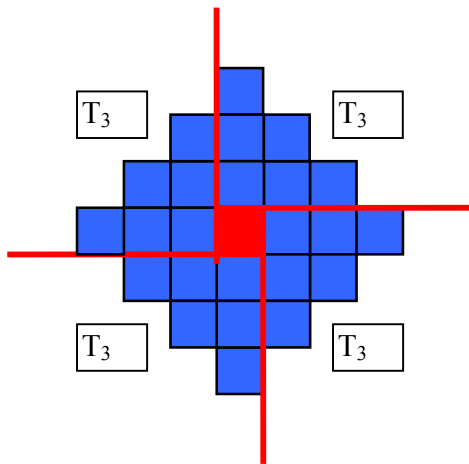


Stage 1 Stage 2 Stage 3 Stage 4

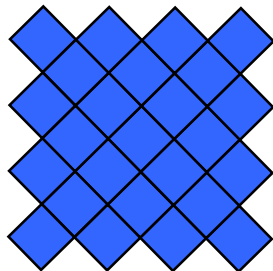
Find a formula for the number of squares (s) at the n th stage.
 Show how you derived it and explain why it works.

This task was set for homework after a lesson on a similar growing pattern of triangles. This lesson established that for t triangles the result of dividing $(t - 1)$ by 3 produced a triangle number. Although pupils were encouraged to tabulate numerical values it was recognised that some pupils would analyse the structure of the pattern.

For the homework assignment one pupil produced the following diagrammatic proof:



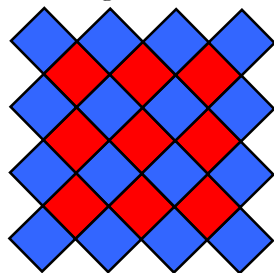
Two pupils produced the following diagram and formula



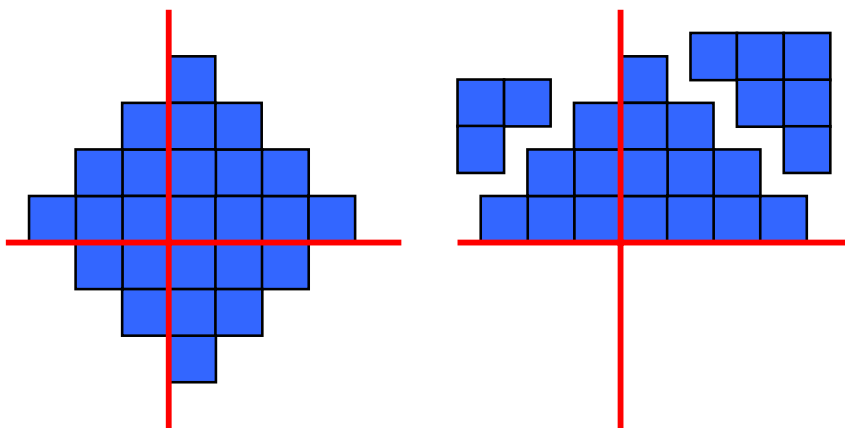
$$s = n^2 + (n - 1)^2$$

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but did not explain the link. Neither they nor their class mates could see the connection when shown the above diagram in the next lesson, until the diagram below was shown to the class.



When this was done the pupils who had produced the original diagram said that it explained what they meant. Most of the pupils in this able group understood that this was a specific example of a result which could be generalised since they could also follow the algebraic proof referred to in the introductory lesson. On another occasion a pupil produced the following diagrammatic analysis of the pattern as the sum of consecutive square numbers.



Exposure to activities of this kind shows pupils that patterns can be explained, and sometimes in more than one way. Exhortation by the teacher that patterns, like all mathematics, should be explained and provision of regular practice helps pupils learn to construct their own, albeit informal, proofs.

Proof methods

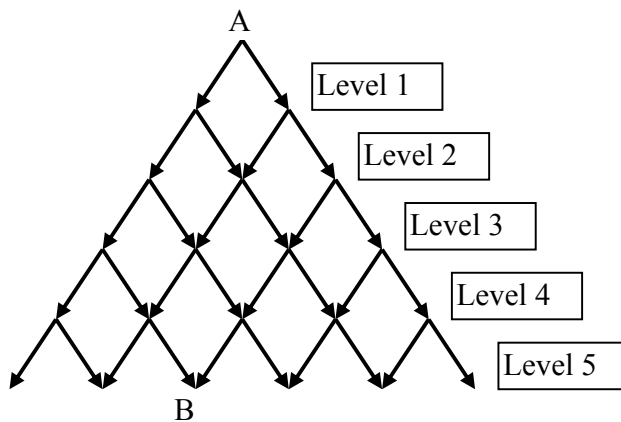
When pupils have been introduced to such proofs of patterns and also some formal traditional proofs it is appropriate to highlight the existence of different proof methods. Another series of lessons hoped to achieve this by considering apparently different problems and using different proof methods to establish confidence in the findings. The mathematics underlying all the problems, namely “Combinations”, has not been studied by the pupils and so they do not have access to rote learned responses.

The first problem – “In how many ways can two colours for a team strip be selected from a range of five colours” – is appropriate for younger children but is used here as an easy

introduction to the main task, and to highlight the method of “proof by exhaustion”. Although younger pupils need guidance in how to list possible outcomes systematically older pupils can do this quickly and thus are sure that there are ten choices. However, the idea that they have used “proof by exhaustion” and the fact that it is appropriate because there are few cases to consider are new.

The second problem – “How many different triangles can be formed by joining three dots in a set of five dots (no three collinear)” – is included because it is mathematically similar, although this is not recognised by pupils. Younger pupils could do this with rubber bands on a pin board but older pupils use pencil and paper. They are expected to understand the link with the previous problem, that choosing three out of five is equivalent to choosing two, and the fact that proof by exhaustion is still feasible.

The third problem – “In how many different ways can you get to each junction of this grid”-



is also based on “Combinations” but, because the context is different, this is not recognised initially by pupils. Even if they do not realise selecting left or right moves from the total number of levels moved this can be made clear in the class discussion about the problem. This discussion highlights the fact that the number of ways of getting from A to B is the same as in the previous two problems because it is equivalent to selecting two right (or left) and three left (or right) moves from five moves. The method of proof used is still “exhaustion” but the question of finding and proving results for a larger grid with, say, twenty levels is raised and the need for a more efficient approach appreciated.

The remaining class discussion returns to selecting colours and elicits the facts that there are five (or n) ways of choosing the first colour and 4 (or $n - 1$) ways of choosing the second. Consequently there are 5×4 (or $n(n - 1)$) ways of selecting a first and second colour and therefore $20/2$ (or $\frac{1}{2} n(n - 1)$) ways of selecting two colours in either order. Pupils can then be reminded that this is a deductive proof and is more powerful than the proofs by exhaustion because the result can be applied with confidence to the problem of selecting two objects from any number of objects, however large.

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The last problem in the series considers the problem of expanding binomials with powers up to five. Introductory class discussion establishes that $(a + b)^0 = 1$ and that the coefficients of a and b in $(a + b)^1$ are both 1 and then reminds pupils of the expansion for $(a + b)^2$. Some pupils need reminding that $(a + b)^3$ can be found by multiplying $(a + b)^2$ by $(a + b)$ and some also need help in starting this. They are then given time to expand $(a + b)^4$ and $(a + b)^5$ and told to examine the coefficients. The summarising discussion establishes that the coefficients are the same numbers as those at Level 5 of the grid above, explains why this is the case and considers the proof method used.

It has also been drawn to the attention of pupils that the pattern in the grid and also formed by the coefficients of binomial expansion form Pascal's Triangle. Pupils are interested in this and can usually see how it can be extended but combinations of more than three from more than five are not discussed at this stage.

To broaden the experience of pupils in mathematics and to introduce a new method of proof, Euler's theorem has been established and its proof discussed with classes of older, able pupils. This involves pupils thinking in three dimensions, provides valuable experience in a previously unfamiliar area of mathematics and a different style of reasoning, and also an unusual example of how changing a problem facilitates its solution. Pupils are given access to a variety of solids and told to count faces (F), vertices (V) and edges (E) and tabulate these in groups – prisms, pyramids and "others". Some pupils are able to find a correct relationship between F, V and E and most recognise that one form is $F + V - E = 2$.

Many able pupils can explain why this is true for a prism whose cross-section is a polygon with n sides – there are $n + 2$ faces (n along the length and one at each end); $2n$ vertices (n at each end) and $3n$ edges (n at each end and a set of n along the length); combining these as $n + 2 + 2n - 3n$ gives the required result of 2. Some able pupils can also analyse pyramids in a similar way to give $(n + 1) + (n + 1) - 2n = 2$ and most can understand this proof. The fact that both these are deductive proofs is highlighted. Pupils enjoy handling the many solids and meeting new words like "parallelepiped", "trapezoidal prism" and "icosahedron" and so have a positive attitude to this activity.

The proof for other solids cannot be proved in the same way, by deduction. The method used explains how any solid can be transformed into a network, by "squashing" it so that faces, vertices and edges become regions (R), nodes (N) and arcs (A) and then derives the proof that $R + N - A = 2$, through class discussion. The proof is by induction and establishes that $R + N - A = 2$ for the simplest case (two nodes joined by an arc); that $R + N - A$ remains constant if an arc or node is added; and that therefore $R + N - A$ must always be two.

In a group of pupils aged 14-15 years and between the 10th and 20th percentiles of mathematical ability numerical results were collated and the relationship $F + V - E = 2$ discovered and also the need for proof recognised. Pupils took an active part in the class discussions about proofs and were given a printed summary. On a later occasion older (aged 15-16 years), very able (at or above 10th percentile) pupils were instructed to investigate polyhedra and networks simultaneously and search for and prove any connections. Their reports on the first stage of the investigation included proofs of the theorem for prisms and pyramids with little or no teacher intervention. After the discussion of the inductive proof for other solids their written

explanations indicated that they understood the proof of the relationship for networks and the equivalence of the relationship for solids.

Conclusion

Although much of the material outlined above has not been specified in the curriculum these wider and deeper mathematical experiences seem to interest and motivate able pupils so that many of them elect to study mathematics further. I have every confidence that their exposure to proof has given them a firm foundation for more advanced mathematics.

References

- Beevers, B. (1994) "Patterns which aren't" *Mathematics in School*, **23**, (5) 10-12
Skemp, R.R. (1971) "*The Psychology of Learning Mathematics*" Pelican
Waring S. (2000) "*Can you prove it? Developing concepts of proof in primary and secondary schools*" The Mathematical Association

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