

Looking back at the beginning: Critical thinking in solving unrealistic problems

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Abstract

We believe that problem-solving skills engage critical thinking at every phase of problem solution. In this research a special attention is given to the first phase - "understanding the problem". We consider this phase as a continuation of all the previous mathematical experience, in which understanding of new problems requires "looking back" at those solved in the past. Evaluation of the givens in the problem sometimes allows immediate solution whereas in other cases it shows that solution does not exist. We found that it is not easy for mathematics teachers to discover that a problem includes contradictory (i.e. unrealistic) conditions. We suggest that such problems should be included into teachers' professional development programs to develop teachers' awareness of the importance of mathematical accuracy and connectedness.

Keywords: Algebraic and geometric tasks; Critical thinking; Problem solving; Polya style heuristics; Teachers professional development

1. The background

1.1. On the importance of unrealistic tasks

In his extensive study on students' mathematical abilities Krutetskii (1976) included *unrealistic problems*, i.e., those that include contradictory givens, as one of the types of problems that allow examination of understanding of mathematical material learned by a student, "which shows up in its processing and retention". An example of such a problem is the following:

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Task 1: *What is the area of an isosceles right triangle with leg equal to $5a$ cm and hypotenuse to $12a$ cm?* (ibid. p. 133)

It is clear that such a triangle does not exist. The given measures of the sides of the triangle and its type (isosceles right triangle) are contradictory. Krutetskii assumed that solvers who solve those problems out of specific context would not be able to realize the unrealistic character of the situation described by the problem.

Our interest in teachers' solving unrealistic problems is based on our belief that identification of unrealistic problems is an integral part of teachers' mathematical knowledge: Understanding of the unrealistic conditions of a problem demonstrates connectedness and consistency of ones mathematical knowledge while identification of inconsistent data when solving a problem demonstrates person's critical thinking. From the pedagogical point of view the ability to identify unrealistic situations may prevent generation of ill-defined examples in the course of mathematics lessons as well as strengthen teachers' critical view of textbooks and other instruction materials.

This research was motivated by our observation that shows that pre-service teachers usually do not identify unrealistic conditions of a problem. It is supported by the analysis of teachers' performance on sorting conditional statements task (Zaslavsky and Leikin, 2004). Zaslavsky & Leikin demonstrated that teachers usually relied on external features of the equations and inequalities (e.g., types of functions, types of conditional statement), and they seldom considered the internal features – the domain and the range of the functions. For example, equation $\log(4-x) - \log(x-6) = 1$ (task 2) does not have a solution (the solution is an empty set) since domains of the two functions are disjoint sets. Thus, there is no need to perform algebraic manipulations in order to find a solution; the result is immediate. Similarly no manipulative solution is needed for the inequality $\sqrt{x^2 + px + 1} < 0$ (task 3): it does not have a solution since radical denotes "arithmetic square root" which is non-negative for any non-negative value of the function under the radical. To solve these tasks shortly, one has not only to think about algorithms of solutions of these problems but also to see the whole context, to connect all the pieces of knowledge related to the task.

There is a similarity in approaching the tasks used by Zaslavsky and Leikin (2004) and unrealistic tasks offered by Krutetskii (1976). Subjects with procedural understanding (Skemp, 1976) of the topic will approach the tasks algorithmically by applying formulas (e.g., task 1 – formula of the area of right triangle) and manipulations (e.g., for task 2 and task 3). Those with relational understanding will consider connections between the given mathematical objects, their properties and components and will conclude that solution does not exist.

1.2. On the cyclic nature of reflection on a solution and understanding a problem

Polya (1973) highlighted four main phases of problem-solving process: understanding the problem, devising a plan, carrying out the plan and looking back at the completed solution. He recommended checking the results and checking the argument in order to make sure that it is correct.

By looking back at the completed solution, by reconsidering and reexamining the results and the path that led to it, they [students] could consolidate their knowledge and develop their ability to solve problems. (Polya, 1973, pp. 14-15)

We argue that in the case of unrealistic problems "looking back" should start at the planning stage of the solution. On the one hand careful analysis of the givens in the problem may outline a solution plan, on the other hand, it may reveal discrepancy of the givens and immediately show that the problem does not have a solution. As noted above we believe that critical thinking is one of the basic cognitive skills supporting and encouraging solution checking.

1.3. On critical thinking in mathematical problem solving

NCTM *Curriculum and Evaluation Standards* (1989) pointed out that "A climate should be established in the classroom that places *critical thinking* at the heart of instruction ... To give students access to mathematics as a powerful way of making sense of the world, it is essential that an emphasis on *reasoning* pervades all mathematical activity." (ibid. p. 25). Erroneously unrealistic problems may be seen as those that prepare students to real life through developing their critical thinking. The ability to think critically is essential if individuals are to live, work, and function effectively in our changing society.

Critical thinking includes the use of cognitive skills or strategies directed at desirable outcomes of human activities of different kinds: solving problems, formulating inferences, calculating likelihoods, and making decisions. It also includes using skills that are effective for the particular context and type of thinking task (Halpern, 1998).

Critical thinking is a mental process of analyzing or evaluating information, particularly statements or propositions that are offered as true. It is a process of reflecting upon the meaning of statements, examining the offered evidence and reasoning, and forming judgments about the facts. Such information may be gathered from observation, experience, reasoning, or communication. Critical thinking has its basis in intellectual values that go beyond subject matter divisions and include: clarity, accuracy, precision, evidence, thoroughness and fairness (Wikipedia, 2005).

Ferrett (2002) included among other attributes of critical thinking the following: assessment of statements and arguments, admitting a lack of information, ability to clearly define a set of criteria for analyzing ideas, examining problems closely, being able to reject information that is incorrect or irrelevant. Critical thinking involves evaluation of the thinking process - the reasoning that went into the conclusion we arrived at.

2. The Investigation

2.1. The purpose

This investigation was aimed at exploring teachers' mathematical performance on unrealistic problems, critical reasoning associated with unrealistic tasks. In particular we examined whether teachers understand the contradictory nature of conditions given in the problem. We also analyzed teachers' views on such kind of tasks.

2.2. Population

We assumed that ET's may succeed better in unrealistic tasks both because of their teaching experience and of their educational background. Thus the population of our study included three groups of mathematics teachers as follows: Seventeen pre-service mathematics teachers (PT) and

48 experienced high school teachers (ET) from two groups participated in our study. PT's had BA in mathematics and were learning for teaching certificate. ETs' experience varied from 5 years to 28 years. Most of the ET's had MA in mathematics or mathematics education. These teachers participated in the study in two groups: ET1 included 27 teachers and ET2 included 21 teachers.

2.3. The instrument

The teacher were asked to complete a written questionnaire. We assumed that conditional formulation of the tasks might evoke teachers' critical thinking. Thus the tasks were formulated in two versions. In one (*non-conditional*) version we asked the teachers to "solve problems". In other (*conditional*) version we asked the teachers to "solve problems if possible". PT's and ET's from one group (ET1) were presented with a non-conditional questionnaire while ET's from group ET2 were presented with a conditional questionnaire.

Our questionnaire included algebraic and geometric problems. Figure 2 shows the tasks presented to the subjects. It also describes correct, alternative and incorrect solutions to the tasks.

Algebraic task required from the teachers "to find sum of the squares of the real roots of the equations without calculating the roots" (A1: "find"; A2 "find if possible"). The teachers were given two equations that did not have real roots. As an integral part of the solution the teachers had to check whether the roots exist. There was no need to perform algebraic manipulations since the equation does not have real roots.

Task Ab had an additional control level: When missing the contradiction in the question at the beginning of the solution, one could find that the sum of the squares of the two numbers is negative ($\alpha^2 + \beta^2 = -\frac{3}{4}a^2 - 1$) and claim that the question does not have answer on the set of real numbers. Alternatively teachers could state that they found sum of the squares of complex roots of the equation.

Geometry task required from the teachers finding area of a right triangle according to its hypotenuse and altitude to the hypotenuse (G1: "find"; G2 "find if possible"). In these two tasks the length of the altitude was bigger than half of the side, thus these measures were inconsistent with the following property of a right triangle: altitude of a right triangle is not bigger than half of the side (see Figure 2). The solution was very simple both when noticing the contradiction and when missing it. Note that all the teachers in the sample group were familiar with the property.

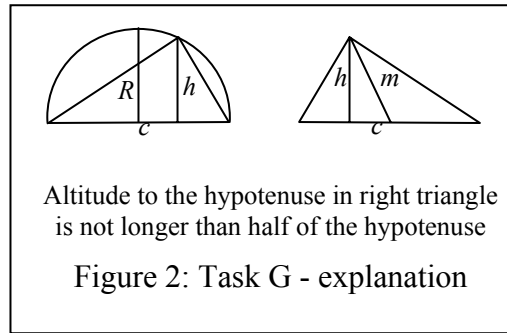
After performing the written assignment teachers checked and corrected their works. They did it in a different color so we could keep track of their initial solutions. Each session was concluded with a whole-group discussion. The discussion in group PT1 was video-recorded and transcribed, discussions with ET's were recorded in writing.

The task	Correct (content-connected) solution	Incorrect (algorithmic) solution	Alternative solution
A1* : Without calculating roots of the equations (α and β) find* sum of the squares of the real roots ($\alpha^2 + \beta^2$), for each one of the following equations:			
Aa. $x^2 - 5x + 7 = 0$	<u>No solution:</u> $\Delta = 25 - 28 < 0$ \Rightarrow no real α and β .	$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha \cdot \beta =$ $= (-5)^2 - 2 \cdot 7 = 11$	$\Delta < 0 \Rightarrow \alpha$ and β are complex numbers $\alpha^2 + \beta^2 = (-5)^2 - 2 \cdot 7 = 11$
Ab. $2x^2 - ax + a^2 + 1 = 0$	<u>No solution:</u> $\Delta = a^2 - 8a^2 - 8 < 0$ \Rightarrow no real α and β	$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha \cdot \beta =$ $= \left(\frac{a}{2}\right)^2 - 2 \cdot \frac{a^2 + 1}{2} = -\frac{3}{4}a^2 - 1$	$\alpha^2 + \beta^2 = -\frac{3}{4}a^2 - 1$ $\alpha^2 + \beta^2 < 0$: impossible for real α and $\beta \Rightarrow$ 1) no solution 2) α and β are complex numbers
G1* : Find*area of the right angle triangles in each one of the following cases:			
Ga. Hypotenuse is $\sqrt{11}+1$ cm, the altitude to the hypotenuse is $\sqrt{11}-1$ cm.	<u>No solution:</u> Altitude to the hypotenuse in right triangle is not longer than half of the hypotenuse	$S = \frac{ch}{2}$ $S = \frac{(\sqrt{11}+1)(\sqrt{11}-1)}{2} = 5$	The condition that the triangle is right angled is surplus If not right triangle: $S = 5$
Gb. Hypotenuse is $2a-1$ cm, the altitude to the hypotenuse is a cm.		$S = \frac{ch}{2} = \frac{a \cdot (2a-1)}{2}$	If not right triangle: $S = \frac{a \cdot (2a-1)}{2}$
* Questionnaires A1 and G1 included non-conditional requirement "find". Questionnaires A2 and G2 included conditional requirement "find if possible"			

Figure 1: The tasks in the questionnaires

3. Solving the tasks

For both versions of the tasks majority of the teachers produced algorithmic (wrong) solutions. Only one ET identified contradiction in the geometric task. All other teachers in both versions of Task G calculated area of the triangles using formula. Teachers succeeded better to some extent in solving the algebraic tasks (see Table 1).



On the non-conditional task, 4 of 17 PT's and 2 of 29 ET's examined delta both in tasks Aa and Ab and concluded that the tasks do not have a solution. In task Ab, which included additional control level, 2 PT's and 6 ET's decided that the task cannot be solved since a sum of the squares of real numbers cannot be negative. When solving the conditional tasks 5 of 21 teachers examined delta at the beginning of the solution and did not perform algebraic manipulations.

Group of teachers	PT (N=17)	ET1(N=29)	ET2 (N=21)
Task			
Aa	Non-conditional	4($\Delta < 0$) 23.5%	2($\Delta < 0$) 7%
	Conditional		5($\Delta < 0$) 28%
Ab	Non-conditional	4($\Delta < 0$) & 2($\alpha^2 + \beta^2 < 0$) 35%	2($\Delta < 0$) & 6($\alpha^2 + \beta^2 < 0$) 27%
	Conditional		5($\Delta < 0$) 28%

Table 1: Teachers performance on Algebraic Tasks

As we expected, conditional task Aa was somewhat easier for the teachers than non-conditional one (Task Aa: 7% of teachers in group ET1 vs. 28% in ET2, Table 1). This tendency was also clear in task Ab at the planning phase of solution (Task Ab: 7% [$2(\Delta < 0)$] ET1 vs. 28% [$5(\Delta < 0)$] of ET2, Table 1). Then again additional control tool [$\alpha^2 + \beta^2 < 0$] helped 6 ET's realize there was no solution for non-conditional Task Aa. Surprisingly, the conditional task, negative value of $\alpha^2 + \beta^2$, did not help teachers who failed to realize that the equation did not have real roots at the beginning of the solution (by checking value of Δ). We assumed that conditional formulation of the problem could evoke critical reasoning to the same extent as the second level of control in Task Ab.

Opposite to expectations ET's did not appear to be more successful in performing the tasks than pre-service mathematics teachers. As a result of the experiment and subsequent group

discussions we have come to the conclusion that the fact that PT's are not less successful in solving this kind of tasks has a reasonable explanation that we outline in the next section of this paper.

4. Discussion of Tasks and Solutions

In the course of the whole group discussions all the teachers who had not realized contradictions in the problems conditions were upset. Some of them were annoyed by the fact that they had missed the inconsistencies, and some were even angry with us for presenting "such unfair tasks". However, majority of the teachers reported that they had enjoyed the experience. In each group there was a disagreement on the "unfairness" of the problems. Only a small number of teachers thought their experience was bad whereas most of them claimed that it was very positive.

There was clear distinction in the teachers' attitude to Task A and Task B. The teachers agreed that Task A was reasonable and "reminded them about the necessity of mathematical accuracy" since "any solution related to quadratic equation should start with examination of delta". One of the teachers said:

How could I miss this? Finding delta when solving the tasks related to the quadratic equation is a part of the algorithm. It is a regular procedure. I always tell my students: "First check whether the equation has real roots, then find the roots, or do whatever the task requires". I just did not think [about delta]. How can you talk about something when you do not know whether it exists.

We agree with the teachers that Task A in our study was less provocative and more regular than Task G. Based on the teachers' reactions in the discussion, we think that performance on algebraic task reflects mathematical culture of their classrooms where under the pressure of time teachers sometimes do not require from their students precise and accurate mathematical performance, where algebraic manipulation and procedures are in the heart of the instructional processes. Similarly to Task 1 ($\log(4-x) - \log(x-6) = 1$) accuracy as characteristic of critical reasoning and relational perspective, allows to shorten the procedure, even not perform it at all. Note that slightly better performance of PT's than of ET's on Task A1 we address to the classroom routines in which ET's are involved every day while PT's are still learning and more challenged by the courses in which they participate.

Task G presented to the teachers was found more "tricky". At the beginning of the discussion, the teachers felt they had never met such kind of tasks before. Though after discussing the task they made an analogy between Task B and other "tasks from the textbooks that include mistakes". Contrary to Task A, which teachers saw as "pretty regular for mathematics classes" and for which wrong solutions they considered "just the result of a mistake", Task G was considered by them inapplicable for the classroom situation. They saw this task as mistaken and claimed that teachers' duty is to avoid such tasks in classroom activities. In teachers' opinion, using textbooks is and should be "safe" and "the authors have to check many times the problems and the solutions that they include in the textbooks". The teachers were certain that problems of this kind confuse pupils. On the other hand, they agreed the task is good for teachers as it

requires thinking about mathematical connections, i.e., "other theorems related to the task, not only those you need for the solution of a specific problem".

During the discussion with PT's the distinction between solving proof tasks from the books and exploring conjectures raised in the course of an inquiry-based lesson was mentioned. Conjectures and hypotheses may be unrealistic and the inquiry procedure has to verify their realistic nature. Refuting a conjecture at the proof stage of inquiry is a natural procedure whereas when meeting a proof or computational task teachers presume that those who ask to prove or find something have already checked that this task is realistic.

5. Concluding remark

We finish this experiment with many open questions. Some of them are raised from our communications with the teachers and their replies while others raised from the mathematical analysis of the tasks we performed in the course of the study. Among other questions we ask: How can we formulate the tasks so that teachers' critical thinking will be evoked? We find Task 1 more transparent than Task G. How different will be teachers' solutions of Task 1 and Task G? How different or similar will be teachers reasoning associated with unrealistic problems and problems with consistent surplus conditions (See, for example, Krutetskii (1976)?

References

- Ferrett, S. (2002) *Peak performance: Success in college and beyond*. McGraw-Hill Clooge.
- Halpern, D. F. 1998. Teaching critical thinking for transfer across domains. *American Psychologist*. 53, 449–55.
- Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. J. Kilpatrick & I. Wirszup: Eds., J. Teller: Translation from Russian. Chicago: The University of Chicago Press.
- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
- Polya, G. (1973). *How to solve it*. Princeton, NJ: Princeton University Press.
- Schoenfeld, A. H. (1986). On having and using geometric knowledge. In J. Hiebert (Ed.). *Conceptual and Procedural Knowledge: The Case of Mathematics*, (pp. 225-64). Hillsdale, N.J: Erlbaum.
- Skemp, R. R. (1987). *The psychology of learning mathematics*. Hillsdale, NJ: Erlbaum.
- Wikipedia (2005). The free encyclopedia.
http://en.wikipedia.org/wiki/Critical_thinking. (downloaded: Dec. 2005)
- Zaslavsky, O. & Leikin, R. (2004). Professional development of mathematics teacher-educators: Growth through practice. *Journal of Mathematics Teacher Education*, 7, 5-32.