

Lagrange: A Well-Behaved Function

Benjamin Harris
The University of Montana*

Abstract

This paper outlines the biography and achievements of Joseph Louis Lagrange (1736–1813) and includes a detailed explanation, with examples, of the Lagrange Multiplier method for optimizing multivariate functions subject to constraint. The Lagrange Multiplier is widely used in chemistry, physics, and economics, in particular. The paper considers the origin of economics' use of the multiplier and provides a concrete example of how it is used in microeconomic theory. While the focus is on the multiplier's application to microeconomics, the intended audience includes all teachers and students who encounter any of Lagrange's contributions. Since Lagrange's contributions to mathematics are numerous, so too are those who might benefit from learning more about the man and his time.

keywords: Lagrange, Lagrange Multiplier, Economics

1 Introduction

Joseph Louis Lagrange (1736–1813) lived during one of the most exciting periods in human history. Newton had published the *Principia Mathematica* less than 50 years, and died less than 10 years, before Lagrange's birth. The *Principia* (1687) not only provided a mathematical framework for analyzing physical phenomena, but it also signaled that the natural world functions according to a set of immutable laws. Insofar as Newton's audience adopted this message, the *Principia* set into motion a widely accepted belief that humans need only learn these laws in order to fully understand the natural world. "The eighteenth century," therefore, "was an age in which the power of the human mind seemed unlimited" (Grabier 1990). This positivist world view was adopted and expounded by mathematicians and philosophers such as Votaire, Kant, Euler, Laplace, Goethe, and others. Eighteenth century mathematicians, "almost without thinking", manipulated the calculus that Newton and Leibniz introduced and pushed it to its limits "without careful attention to convergence of series, without knowing under what conditions one might change the order of taking limits, [without well-founded knowledge] of integration..." (*ibid.*). Grabier (1990) further notes that

*Department of Economics, Liberal Arts 414, Missoula, Montana 59801,
e-mail: benjamin.harris@umontana.edu

physicists had been so successful with application of mathematics in explaining the natural world that they did not stop to show *why* mathematics was appropriate for describing the physical universe with a rigor that did not appeal to intuition.

Lagrange, no doubt, shared the optimism about mathematics and human progress that characterized his time. He chose the role, however, of the skeptic. His primary objective in mathematics—and life, it appears—was not to add to the list of new applications of calculus, but rather to revisit its foundations and offer a more rigorous explanation of how and why calculus works (Grabier (1990); Sarton (1944)). Before Lagrange, no formal definition of limits existed. Instead, the idea was explained by the *ad hoc* concepts of infinity and infinitesimals rather than by algebraic expressions of real numbers. Lagrange believed that conceptual or intuitive explanations had no place in a truly rigorous subject, and so Lagrange endeavored to reduce the foundations of calculus to algebra, which had already been established as a dependable representation of the real world.¹

The purpose of this paper is to give an introduction to Lagrange’s life, work, and influence on contemporary explanations and applications of calculus. Those who teach Lagrange’s methods in calculus and applied sciences will find this a useful supplement to enrich lectures or to assign to students. Those in the economics profession, in particular, will benefit from this paper since every intermediate microeconomics student must be intimately familiar with one of Lagrange’s techniques, the Lagrange multiplier. Few economists, however, learn about the multiplier’s origins. In the spirit of its creator, this paper seeks to transform merely a useful tool into an rigorously understood science. The discussion, therefore, will proceed as follows: Section 2 will present Lagrange’s biography, using the various cities in which he lived as a geographical framework for the phases of his intellectual growth. Section 3 will describe some of the contributions Lagrange made to mathematics and other disciplines, and it will give a specific example of how theoretical microeconomists regularly utilize his work. Section 4 will conclude.

2 Biography

2.1 Piedmont (1736–1755)

Lagrange was born in Piedmont, northern Italy, to where his great-grandfather had immigrated from Touraine, France in order to direct the Treasury of Construction and Fortifications under Charles Emanuel II Duke of Savoy.² His family was apparently wealthy, although much of that wealth was lost to failed financial speculation during his boyhood.

¹Geometry had long been accepted as an appropriate tool for describing and analyzing natural phenomena. Descartes’ work demonstrated the relationship between algebra and geometry and, by extension, the relationship between algebra and the natural world. Lagrange recognized, then, that by linking calculus to algebra, he would authenticate its use in describing the physical world.

²While his great-grandfather and name came from France, Lagrange can hardly be classified as French, himself. His grandfather and father both married Italian women, and Lagrange was raised in Italy. Nevertheless, we cannot so easily classify Lagrange as an Italian, either. Early in his life, he relocated to Berlin and then later moved to Paris. Additionally, Lagrange composed practically all his scholarship in French.

During his early childhood, Lagrange's interests were more classical than scientific (Simons (1972)), but he became enthusiastic about mathematics after reading a paper by Edmund Halley on the use of algebra in optics (1693). Sarton (1944) writes that Lagrange thereafter mastered "...with incredible speed the works of Newton, Leibniz, Euler, the Bernouli, [and] d' Alembert." By the time he was 18 years old, Lagrange began independent research into outstanding mathematical problems and established correspondence with Euler and Fagnano. Lagrange's first paper, which initially appeared as a letter to Fagnano from July 1754, drew an analogy between the binomial theorem and the series of derivatives of the product of functions. He also sent his findings to Euler, who was especially impressed with the young Lagrange, whose discovery of "a new series for the differentials and integrals of every grade corresponding to the Newtonian series for powers and roots" was spoiled only by the chance fact that Leibniz happened to have published the same discovery 50 years earlier. Lagrange's fear of being labeled a plagiarist fueled further efforts to make scientific contributions of real value. He began to work on finding solutions for the maxima and minima of the tautochrone, a curve on which a weighted particle will always arrive at a fixed point at the same time, regardless of its initial position (Chiang (1984); Koetsier (1986)). Lagrange's talent was apparent. The following year (1755), The Royal School of Mathematics and Artillery invited the him to teach in Turin.

2.2 Turin (1755–1766)

Lagrange continued to nurture his relationships with Euler, Fagnano, and d'Alembert during his tenure at the Royal School. During the first year there, Lagrange made up for the embarrassment of his first paper by helping Euler both solve some of the isoperimetric problems that had been plaguing him and also develop the calculus of variations. Euler, like Lagrange after him, was concerned with explaining calculus in algebraic and analytical terms: that is, without using intuition or the metaphor of geometry. In his 1744 classic, *A Method for Finding Curved Lines Enjoying Properties of Maxima or Minima*, Euler lamented that the solution to one of the problems considered in the book not purely analytical, writing "...we have no method that is independent of the geometrical solution..." (Koetsier (1986)). Lagrange soon thereafter presented Euler, who was 30 years older than Lagrange, with a purely analytic solution. Euler was so pleased that he wrote to his junior colleague, "...I am not able to admire you enough..." (Grabier (1990); Sarton (1944)).

Around 1759, Lagrange collaborated with the chemist, Saluzzo de Monesioglio, and the anatomist, Gian Francesco Cigna, to found the Turin Academy of Sciences. Academies provided an environment in which well-funded research could proceed unimpeded by the distraction of teaching. Accordingly, the Turin Academy quickly began to publish journals, the first volume of which contained 3 papers by Lagrange (Sarton (1944)). These papers featured his research on the nature and propagation of sound, the movement of the moon, and on the satellites of Jupiter. Meanwhile, he published a paper in the *Mémoires de l'Académie de Berlin* on an analytical expression of tautochronous curves. These papers earned Lagrange both prizes from the Parisian Academy as well as recognition throughout Europe. By 1766, Lagrange was ready to leave Turin, and his friend, Euler, played a major

role in ushering Lagrange to the next phase in his life.

2.3 Berlin (1766–1787)

Leibniz founded the Prussian Academy in 1700; and under the management of Frederick II (Frederick the Great), it became one of the premier academies in Europe, housing people like Euler, d’Alembert, Lambert, Kant, and Diderot. Euler left his position as Director of the Mathematical Section at the Prussian Academy for St. Petersburg in early 1766, and Lagrange assumed that post by the end of the year with the recommendations Euler (who was leaving) and d’Alembert (who refused the first invitation). The King famously wrote to d’Alembert after attaining Lagrange that he was happy to have “...replaced a one-eyed geometer with a two-eyed one...”³ (Sarton (1944); Koetsier (1986); Grabier (1990)).

Lagrange’s was most productive in Berlin. He worked mostly on mathematical and mechanical problems, and his writings were frequently published by the Academies of Berlin, Turin, and Paris. During the same time, Lagrange wrote a 315 page supplement to the second volume of the French translation of Euler’s *Elémens d’algèbre*. Sarton (1944) describes Lagrange’s time in Berlin as “...the richest years of his life...”.

Despite the admiration that Lagrange’s work afforded him, his peers and mathematical historians characterize him as exceedingly modest. Whereas Lambert famously enumerates the leading mathematicians, “Euler and d’Alembert are first and equal, Lagrange is third, and I am fourth, and there is no need of going any further as I cannot think of anybody else worth quoting after these”, Lagrange was much more humble. His first and almost ritualistic response to any question was always “*Je ne sais pas*”⁴. Lagrange was diffident when presenting new ideas to friends like d’Alembert. Sarton (1944) quotes Lagrange as having bemoaned the fact that he could not present d’Alembert with something better than the *Mécanique Analytique* to win the latter’s favor. Sarton (1944) points out that Lagrange’s diffidence went beyond “...the conventional exaggeration which eighteenth-century courtesy required.”

During his 21 years in Berlin, Lagrange developed the reputation of being somewhat 1-dimensional. That is, he seems to have been primarily—even solely—concerned with conducting mathematical research. When d’Alembert wrote to congratulate Lagrange on his first marriage and the happiness he was certain to derive from it, Lagrange responded that he had no taste for marriage, and would have avoided it altogether if not for his need of a nurse and housekeeper to provide him with the quiet conveniences necessary for a life of research (Sarton (1944)). Lagrange had no passion for musical performances, except that they provided a nice environment in which he could ponder mathematical problems without interruption. His daily diet and routine were unconventional and strict, and each day included at least 8 hours of uninterrupted and isolated study. Lagrange did not have many close acquaintances.

³Euler and Frederick did not have particular fondness for one another, and the King is here poking fun at Euler’s loss of sight that was just beginning. He was completely blind at the time of his death.

⁴“I do not know.”

Lagrange's sacrifice of companionship and his focus paid off. While in Berlin, he composed most of the *Mécanique Analytique*. His masterpiece, however, was published after Lagrange arrived in Paris. When Frederick the Great, to whom Lagrange was *persona grata*, died, the latter knew his position at the Prussian Academy would soon either end or become remarkably less agreeable under the new direction of Frederick William II. The French government offered to install Lagrange at the Paris Academy, and he quickly accepted. He left Berlin on 18 May, 1787 (Sarton (1944)).

2.4 Paris (1787–1813)

In Paris, Lagrange completed, edited, and published his magnum opus, the *Mécanique Analytique* (1788). Many of Lagrange's time and afterward welcomed the work with enthusiasm. Ernst Mach called it "...a stupendous contribution to the economy of thought..." (Grabier (1990)), Vittorio Fossombroni called it "immortal" (Sarton (1944)), and William Rowan Hamilton described the *Mécanique Analytique* as "a kind of scientific poem" (Simons (1972)). In the book, Lagrange distilled classical mechanics to the calculus of variations, which he then further reduced to pure algebraic analysis. Koetsier (1986) quotes Lagrange as writing in an advertisement for the book: "There are no figures at all in this work. The methods that I demonstrate require neither constructions nor geometrical or mechanical reasoning, but only algebraic operations subjected to a regular and uniform procedure. Those who love analysis will be pleased to see that mechanics is becoming a new branch of it and they will be grateful to me for having thus extended its domain."

After the publication of the *Mécanique Analytique*, Sarton (1944) writes, Lagrange was mentally exhausted and his contributions lessened. His inactivity could also be due, in part, to his being distracted by his work at the newly formed Commission of Weights and Measures, and also to the stress caused by the French Revolution, which began 1 year later. He had been invited to France by its pre-Revolutionary government, had stayed at the Louvre, and could easily have been marked as an enemy to the revolution during the Terror (1793–1794). Most of his colleagues were removed from their positions on the Commission, and Lavoisier was put to death. Lagrange, however, was retained, probably due to his characteristic modesty and aversion to conflict.⁵ During his position on the Commission, Lagrange promoted the official adoption by the French government of the metric system, including the use of decimals.

After the Revolution, he fell into the favor of Napoleon Bonaparte, who enjoyed sharing geometrical puzzles with Lagrange and Laplace. Napoleon's goodwill afforded Lagrange more security and comfort than upheaval of the Revolution. Lagrange spent the first part of the 19th century—and the final part of his life—teaching and revising many of his earlier works. During his last conversation with his colleagues, Lagrange said, "Oh! death is not to be feared and when it comes without pain it is a last 'function' which is neither distressing nor disagreeable...I have had a long career; I have obtained some fame in mathematics. I have never hated anybody, I have done no harm, and there must be an end..." (Sarton (1944); Koetsier (1986)). Lagrange died 10 April, 1813. He lived 77 years.

⁵"I don't know why they kept me", Lagrange said (Sarton (1944)).

3 Lagrange's Mathematical Contributions

Many of the practices of students in modern calculus courses derive from the work of Lagrange. Most generally, Lagrange, with other mathematicians, endeavored to strengthen the foundation of calculus, which had so far rested on geometrical and intuitive arguments. As mentioned above, proving the fundamentals of calculus in algebraic terms lent a more detailed and rigorous understanding of the system, established calculus' place in the canon of accepted branches of mathematics, and paved the way for advances in the science.

Lagrange's contributions to how we understand the limit of a function is an example of how he worked to introduce rigor into calculus. During Lagrange's time, mathematicians understood what we now call the derivative to be the slope of the a function when the change in the input (x) is *infinitesimally* small. *Infinitesimals* make sense to modern calculus students who can compare the concept to the formal definition of the derivative as the limit of the difference quotient as the denominator approaches 0. For Lagrange, however, there was no formal definition to which he could relate the concept of *infinitesimals*. There was no definition of continuity or convergence of sequences. In 1774, Lagrange and his colleagues at the Prussian Academy hosted a contest for advancements in the foundations of calculus (Grabier (1983)). And while Lagrange, himself, never developed satisfactory explanation for the foundations of calculus, he played a major role in creating an atmosphere in which such an explanation was necessary.

His work with inequalities, also, helped Cauchy develop a formal definition of the limit, the modern $\delta - \epsilon$ proof of the derivative, and the definition of the derivative as the limit of the difference quotient (Grabier (1983)). These definitions (and the proofs that accompany them) were, of course, exactly the kind of advancement for which Lagrange was looking. They translate nearly 200 years of intuitive understanding of calculus into a form with the certainty of algebra.

Some of Lagrange's more superficial—but no less important—contributions include the prime notation, whereby for a function, $f(x)$, its derivative is denoted $f'(x)$. Lagrange was the first to use the term, *dérivée*, from which we get the term 'derivative'. He also developed the use of ∂ to indicate a partial derivative in a multivariate function. And, for that matter, he developed the multiplier method for calculating optimizations of one multivariate function that is constrained by another.

3.1 The Lagrangian Multiplier

During his exchange of letters with Euler, Lagrange became inspired by the former's 1744 work, *A Method for Finding Curved Lines Enjoying Properties of Maxima or Minima*. In it, Euler worked to find the extrema of functions that are defined on a set of functions rather than a set of numbers on points (Koetsier (1986)). His solutions to these problems were, however, not purely analytical; and it was Lagrange who provided the following strictly analytical method for solving constrained optimizations.

3.1.1 The General Case

Suppose we wanted to find the optimization of some function $z = f(x, y)$, where the variables x and y are not independent. Instead, they are dependent on the side condition, $g(x, y) = 0$, which we call the constraint. That is, we want to:

$$\max_{x,y} z = f(x, y) \quad \text{subject to} \quad g(x, y) = 0$$

The usual way to solve this problem is to arbitrarily assume 1 of the variables is independent and the other dependent. If choose x to be the independent variable, then we can compute $\frac{dy}{dx}$ from the constraint function:

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \frac{dy}{dx} = 0 \tag{1}$$

which can be rewritten as:

$$\frac{dy}{dx} = -\frac{\partial g / \partial x}{\partial g / \partial y} \tag{2}$$

Now, we can do the same for the function, finding $\frac{dz}{dx}$ and setting it equal to 0 (by definition of the maximum or minimum of the function), such that:

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0 \tag{3}$$

If we plug equation (2) into (3), we get:

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \frac{\partial g / \partial x}{\partial g / \partial y} = 0 \tag{4}$$

If we solve (4) and the constraint function simultaneously, we get the required points (x, y) to optimize z . One problem with this method, however, is that x and y occur symmetrically, but are treated unsymmetrically (Simons (1972)).

Lagrange's method, which Simons (1972) calls more elegant, begins by forming a new function:

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) \tag{5}$$

Since the constraint is now built in to the new function, F , we can treat it like an unconstrained optimization problem (Chiang (1984)). So, we take the partial derivatives of F with respect to x, y and λ and setting them equal to 0:

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \\ \frac{\partial F}{\partial y} &= \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \\ \frac{\partial F}{\partial \lambda} &= g(x, y) = 0 \end{aligned} \tag{6}$$

If we solve $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ for λ and set them equal to each other, we get:

$$-\frac{\partial f/\partial x}{\partial g/\partial x} = -\frac{\partial f/\partial y}{\partial g/\partial y} \quad (7)$$

Which can be written, along with $\frac{\partial F}{\partial \lambda}$, to give:

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \frac{\partial g/\partial x}{\partial g/\partial y} = 0 \quad \text{and} \quad g(x, y) = 0 \quad (8)$$

In this way, Lagrange reached the same conclusion as (4) without introducing the arbitrary assignment of an independent variable or disturbing the symmetry of the problem. Introducing the new variable, λ , which falls out of the problem, is a small expense. The parameter, λ , is called the *Lagrange multiplier*; and economists love it.

3.1.2 The Lagrange Multiplier in Economic Theory

All economic inquiries, at their most basic levels, ask how decision makers can achieve the greatest benefit, given their situation. The benefit can come from a wide range of possibilities, including time spent with family, the number of trees planted in one's yard, or the number of deep dish meat lover's pizzas one eats. But when decision makers make their decisions, they do not do so based only on the amount of the benefit that would satisfy them, but also on the amount of that benefit they can afford. That is, with no constraints, a father might want to spend all of his time with his family; but at the very least there is the time constraint whereby he needs to devote at least a few hours each day to sleep. A homeowner may want to plant hundreds of trees in her yard, but she may only have enough land to support 50 trees. And that dieter may want to eat 3 of those pizzas before lunch is over, but she is constrained by a diet that only affords her 900 calories a day. In short, economists are concerned with maximizing benefit (commonly referred to as "utility") subject to the constraint of limited resources. Lagrange's method provides a simple way of representing these problems. It was first used by Westergaard in 1876, over 100 years after Lagrange developed it (Davidson (1986)).

3.1.3 The Lagrange Multiplier in Economic Theory: an Example

Suppose there were some decision maker, Kelda, who enjoys cooking with xanthan gum and yams; and the pleasure (utility) she derives from cooking with these ingredients is characterized by the function $U = x^{.5} + y^{.5}$, where x represents the amount of xanthan gum she cooks with, and y represents the number of yams she uses. Suppose that Kelda spends all of her income, I , on these two ingredients. Then, the problem of maximizing Kelda's pleasure is given by:

$$\max_{x,y} U = x^{.5} + y^{.5} \quad \text{subject to} \quad P_x x + P_y y = I$$

Where P_x and P_y are the prices of xanthan gum and yams, respectively. We can build a new function, F , to incorporate this information about Kelda's pleasure, such that:

$$F = x^{.5} + y^{.5} + \lambda(I - P_x x - P_y y) \quad (9)$$

Now, we maximize F by taking the partial derivatives with respect to x, y , and λ , so:

$$\begin{aligned} \frac{\partial F}{\partial x} &= .5x^{-.5} - \lambda P_x = 0 \\ \frac{\partial F}{\partial y} &= .5y^{-.5} - \lambda P_y = 0 \\ \frac{\partial F}{\partial \lambda} &= I - P_x x - P_y y = 0 \end{aligned} \quad (10)$$

If we eliminate λ from $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ and solve for x , we get:

$$x = \frac{P_y y}{P_x} \quad (11)$$

If we plug this into the constraint, then we get the optimal number of yams:

$$y^* = \frac{I}{2P_y} \quad (12)$$

and the optimal amount of xanthan gum:

$$x^* = \frac{I}{2P_x} \quad (13)$$

With each of these last 2 equations, we have calculated a mathematical expression of what we intuitively know: namely, that if income stays the same but the price of a good increases, we will buy less of that good, holding all else constant. We feel good for having done such a thing and are grateful to Lagrange for having provided us with a way to do it.

4 Conclusion

One of the greatest pleasures of studying the history of mathematics comes from discovering the strange habits mathematicians had. The Pythagoreans did not eat beans. Descartes never got out of bed before 11 o'clock in the morning (until, that is, he quickly died after being forced to begin his day at 6 AM to tutor queen Christina). And Lagrange was a shy man who kept a very regular schedule, avoided all conflict, answered each question, "Je ne sais pas", and seemed to think only of math. What is so wonderful about learning these traits is that it makes living people of these text book figures whom we would otherwise know only through the mathematical theorems, coordinate systems, and notation styles named after them. That is, learning about the lives of mathematicians allows teachers and students of mathematics to carry their education beyond lifeless memorization and

mechanical manipulation to something wholly more fulfilling. Knowing Lagrange's humanity makes his mathematical achievements somehow more extraordinary and less-easily taken for granted. During his tireless devotion to a goal he never saw achieved, Lagrange—almost as an aside—provided generations with the tools necessary to achieve that goal and many others.

References

- Chiang, Alpha C.**, *Fundamental Methods of Mathematical Economics*, 3 ed., McGraw-Hill International Editions, Economics Series, 1984.
- Davidson, Thorkild**, "Westergaard, Edgeworth and the Use of Lagrange Multipliers in Economics," *The Economic Journal*, September 1986, 96 (383), 808–811.
- Grabier, Judith V.**, "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus," *The American Mathematical Monthly*, 1983, 90 (3), 185–194.
- , *The Calculus as Algebra: J. L. Lagrange, 1736–1813* Harvard Dissertations in the History of Science, Garland Publishing, Inc., 1990.
- Koetsier, T.**, "Joseph Louis Lagrange (1736–1813) : his life, his work and his personality," *Nieuw Archief Voor Wiskunde*, 1986, 4 (3), 191–205.
- Sarton, G.**, "Lagrange's Personality (1736–1813)," *Proceedings of the American Philosophical Society*, 1944, 88, 457–496.
- Simons, George F.**, *Differential Equations: With Applications and Historical Notes*, McGraw-Hill Book Company,