INTEGRATING INTUITION: THE ROLE OF CONCEPT IMAGE AND CONCEPT DEFINITION FOR STUDENTS' LEARNING OF INTEGRAL CALCULUS

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Abstract: *In this paper we analyze students' conceptual learning regarding the notion of the definite integral. By means of a comprehensive questionnaire, students' concept image and concept definition, as mentioned in the model by Tall and Vinner (1981), were ascertained as well as the corresponding problem solving competence. Participants in this study were 24 students in grade 12 of a German secondary school. The results indicate that definitions play a marginal role in students' learning whereby intuition inherent in concept images dominates the conceptual learning. Based on these subjective convictions, intended and realised knowledge may deviate from each other and might cause difficulties for students.*

We use this pre-introduction to establish a relationship between the content of this paper and our supervisor, colleague and friend to whom is dedicated this volume. In his research, Günter Törner has always been guided by his dual viewpoint, on the one hand, as mathematician and, on the other hand, as mathematics educator. He has always connected his profound mathematical background to didactical approaches in a very fruitful and meaningful way. In particular, he has always rejected pure rote learning and instead emphasized the deep understanding of mathematical notions in order to make sense and grasp the meaning of mathematics beyond definitions. This also includes the development of own ideas and rich images. The focus of our paper follows this approach by analyzing students' learning of the definite integral using Tall and Vinner's (1981) model of concept image and concept definition.

1. INTRODUCTION

Already in the last century, the famous mathematician Poincaré (1908/52) described impressively the contrast between the nature of mathematics, on the one hand, and students' difficulties to grasp mathematical ideas, on the other hand, when asking the following:

> How is it that there are so many minds that are incapable of understanding mathematics? Is there not something paradoxical in this? Here is a science which appeals only to the fundamental principles of logic […], and yet there are people who find it obscure, and actually they are the majority. (p. 117)

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In order to explain this phenomenon, students' learning has been studied from different perspectives, focusing especially on subjective conceptions related to the objective mathematical ones. Bauer (1995), for example, elaborates on individual and subjective aspects of students' learning of mathematics and poses the general question, *What is the relation between mathematics and human thinking?* Addressing also findings from cognitive psychology, this more or less philosophical question can be refined as follows, *What possibilities does our cognitive architecture offer, what restraints does it entail?* More concretely, *Which habits do hinder an adequate formation of mathematical concepts?* Everyday concepts cannot always be uniquely characterized; consequently, limits of concepts are fuzzy. In contrast, mathematical concepts are ideal ones, explicitly concretized in formal definitions. As an example for this decisive but not always transparent distinction serves the following illustration:

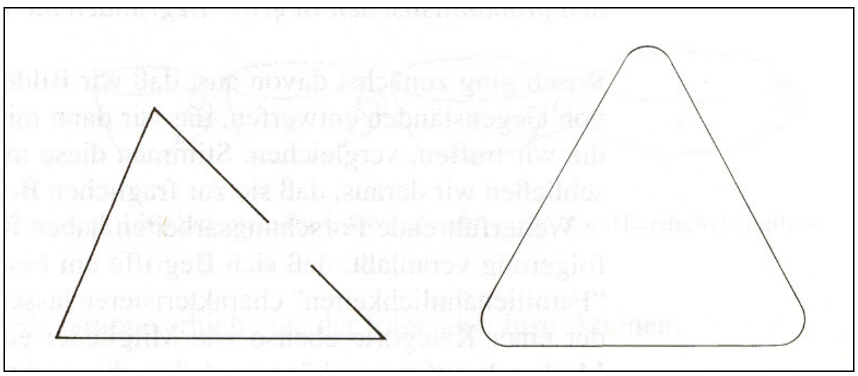


Figure 1.1: Examples for triangles in everyday contexts but not in mathematical ones (Hartland, 1995, p. 217).

The immediate categorization of both figures as triangles describes an essential capacity of our brain, simultaneously constituting difficulties on the formal mathematical level. Hence, mathematical concept formation is different from human thought processes, mostly guided by heuristic strategies (Tversky & Kahneman, 1974). Compared to everyday life, the role of definitions in mathematics is critically raised by Freudenthal's (1973) saying: "Nobody defines what it means to breathe, to walk, to fall, to swim, and nevertheless people learn to do so […]" (p. 317). In the work of mathematicians, however, definitions play an essential role.

The interplay between subjective conceptualizations and formal definitions of mathematical concepts is discussed in this paper by employing Tall and Vinner's (1981) model of concept image and concept definition. We will further elaborate on these dual constructs in the next chapter. Particularly, we will point out how this model can be applied to analyze students' conceptions of a specific mathematical notion, that is, the definite integral.

2. STUDENTS' LEARNING OF MATHEMATICAL CONCEPTS

Students' ways of mathematical thinking, particularly cognitive aspects, have been analyzed from different perspectives and with different emphasis. Nevertheless, these approaches, whether they are more cognitive based (Harel & Sowder, 2005) or more visual oriented (Even & Tirosh, 2002; Presmeg, 2006), share a constructivist view on students' learning and especially refer to students' intuitive thinking as decisive part of acquiring new knowledge. Fischbein (1987) examined students' individual concept formation and elaborated on how intuitive models influence the learning of mathematics:

> It is very well known that concepts and formal statements are very often associated, in a person's mind, with some particular instances. What is usually neglected is the fact that such particular instances may become, for that person, universal representatives of the respective concepts and statements and then acquire the heuristic attributes of models. (p. 149/150)

During their learning of mathematics, students are faced with a wide range of information. How they choose to integrate new mathematical aspects and develop concepts will also depend on their beliefs, values and previous experiences. Consequently, the field of students' subjective concept formation can also be analyzed from the viewpoint of epistemological beliefs (Köller, Baumert & Neubrand, 2000). In this regard, we refer to Schoenfeld's (1998) notion of beliefs as "mental constructs that represent the codifications of people's experiences and understandings" (p. 19). That is, acquiring a mathematical concept is mainly influenced by ontological and epistemological assumptions that have become manifest through students' hitherto learning history.

In particular, Tall and Vinner (1981) have dealt with creative processes in students' learning of mathematics. Their model of concept image and concept definition allows for analyzing students' representations of mathematical concepts. Vinner formulated the terms concept image and concept definition in 1980. Initially, he analyzed how students conceive simple geometric figures and the relations between them (Vinner & Hershkowitz, 1980). The aforementioned terms were introduced in order to categorize students' difficulties emerging because of their tendency to favor prototypical learning. Students focused on typical examples and neglected information given by the mathematical definition. Geometrical figures like a rectangle are represented as typical examples and students therefore have difficulties to consider, for example, a square as a rectangle. The observations made during that study are supported by comparable findings of psychologists, particularly Rosch's (1975) research on prototypes. Contrary to mathematics, concepts in everyday life are learned through examples rather than abstract rules.

At that time, Tall investigated students' cognitive conflicts when learning calculus, in particular when dealing with limits and continuity. The findings of both researchers, Tall and Vinner, led to a common paper in 1981 concerned with concept image and concept definition on which we will elaborate more deeply in the following. The approach presented there has been refined by both authors, partly in separate papers, and also by other researchers within the community and remarkably, the notion is still relevant in the current research (Ouvrier-Buffet, 2006). Another study published at the same time and worth mentioning here, is the one by Cornu (1981) who labels the subjective representations of students as *modèles propres*. His objective has also been to show how these individual concept formations hinder adequate learning of the intended mathematical theory.

2.1 Concept Image and Concept Definition

Tall and Vinner (1981) contrast the cognitive processes of conceptual learning with structural features and particularities of mathematics:

> The human brain is not a purely logical entity. The complex manner in which it functions is often at variance with the logic of mathematics. It is not always pure logic that gives us insight, nor is it chance that causes us to make mistakes. To understand how these processes occur, both successfully and erroneously, we must formulate a distinction between the mathematical concepts as formally defined

and the cognitive processes by which they are conceived. (p. 151)

By using the constructs of concept image and concept definition, Tall and Vinner differentiate aspects of mathematical knowledge that, on the one hand, are given by formal definitions and, on the other hand, by subjective constructions. The figure 2.1 illustrates these two constructs and the ideas around them.

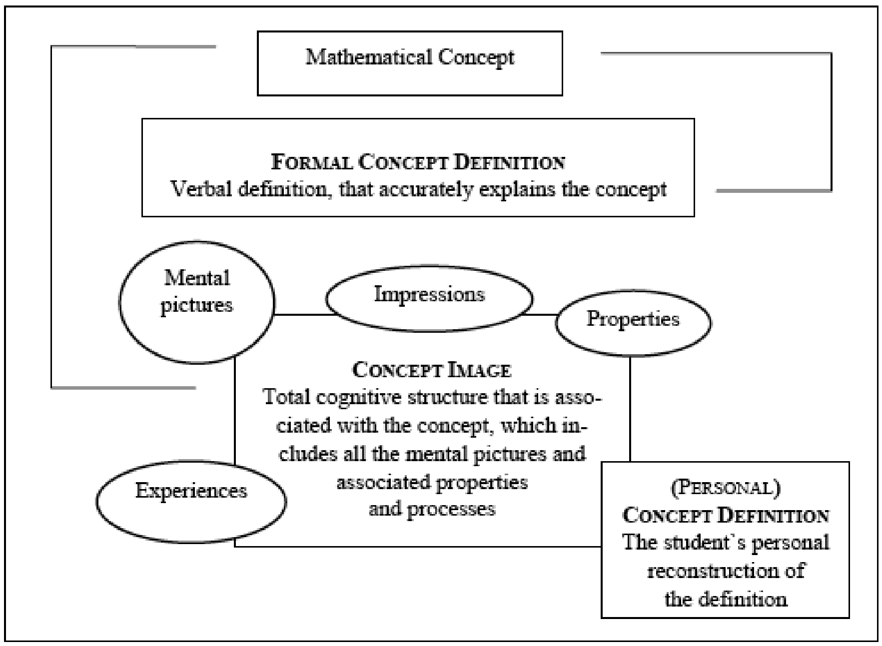


Figure 2.1. Exemplification of concept image and concept definition.

Tall and Vinner (1981) use the term concept image "to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). The concept image may also include a (personal) concept definition as an individual reconstruction of the mathematical one. Further, Tall and Vinner use the term evoked concept image to describe the currently activated part of the concept image. Interestingly, the concept image is not necessarily consistent and coherent regarding its content; it might also include contradictory aspects that students are not aware of.

Finally, we would like to emphasize the importance of this model for learning and teaching mathematics. Explanations for concepts will easily be forgotten if students are not able to develop own ideas and associations. Learning a new concept requires forming a comprehensive concept image but one should keep in mind that maybe important aspects of a mathematical concept are not adequately represented. In the following section we further elaborate on this idea.

2.2 Students' Alternative Conceptions

In order to show the significance of the model, Vinner (1994) contrasts the concept formation in technical contexts with everyday life contexts. Most everyday concepts are acquired as ostensive definitions and, to that effect, formal definitions are of inferior relevance. In everyday contexts, concept images play a crucial role whereas in technical contexts definitions are often essential. Specific to mathematics, it is mostly indispensable to consider all aspects of a definition. For example, if students are asked to examine a continuous (real-valued) function on a closed interval regarding relative extrema they tend to neglect the ones in the endpoints of the interval. In that case students' thinking is dominated by the concept image of horizontal tangent and they do not take into account that the corresponding condition $f'(x) = 0$ is only relevant for the open interval.

By the given model, Vinner (1994) explains why students' misconceptions, like the one mentioned before, occur in learning situations. He shows that during the process of concept formation the relation between the concept image and concept definition is reciprocal (figure 2.2):

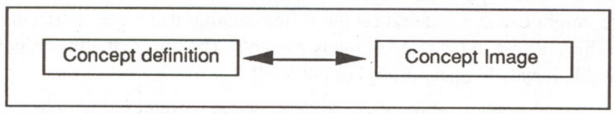


Figure 2.2 Relationship between concept image and concept definition during concept formation (Vinner, 1994, p. 70).

In contrast to that situation, teachers assume a one-way relationship from the concept definition to the concept image, like it is shown in figure 2.3:

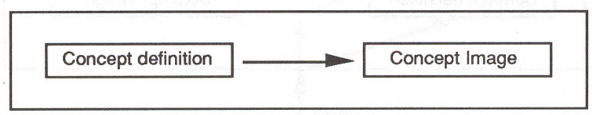


Figure 2.3. Expected relationship between concept image and concept definition during concept formation by teachers (Vinner, 1994, p.70).

The assumption that the concept definition controls the content of the concept image would indicate an adequate concept formation. Vinner & Dreyfus (1989) point out that usually the concept image is not build on definitions but essentially determined by typical examples:

> Hence, the set of mathematical objects considered by the student to be examples of the concept is not necessarily the same as the set of mathematical objects determined by the definition. (p. 356)

In traditional classroom settings, examples are primarily used to introduce a new concept. The learning and later reproduction of a concept is intertwined with diverse conscious as well as unconscious processes and the formed concept image will play a crucial role.

Teachers also expect a comparable one-way relationship from concept definition to concept image during problem solving. However, Vinner (1994) could show that the concept definition does not play any role when students are working on problems:

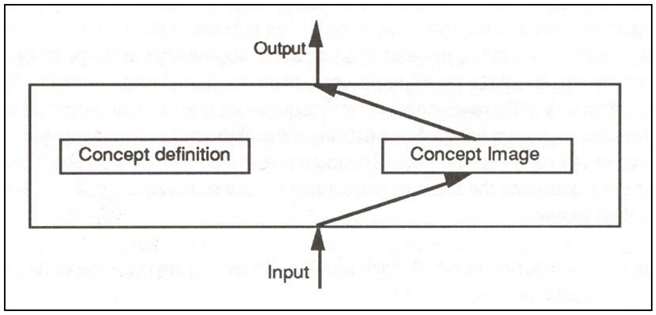


Figure 2.4. Significance of the concept image during problem solving (Vinner, 1994, p. 73).

When working on problems, students do not consider any concept definition. They base their decisions on the concept image. To that effect, Vinner (1994) could show how obstacles in calculus occurred since students remain on a restricted concept image of a tangent, already developed earlier. This concept is usually introduced as tangent on a circle. This concept image provokes difficulties in students' learning of calculus when confronted with a different, namely analytical, definition of a tangent. Additionally, elaborating on secant lines converging on a tangent line reinforces the hitherto developed concept image, which is mostly not enlarged by aspects of a more general tangent. Vinner (1994) found out that, among other things, students had difficulties to draw a tangent having more than one point in common with the curve.

2.3 The Integral Concept

According to the curricula for secondary schools in Germany, the central conceptualizations in calculus should be built on different intuitive ideas supporting each other. Nevertheless, these ideas should not be restricted to the area aspect but substantiated by also focusing on applications. The favored approach to integral calculus in German schools is linked to the Riemann integral without defining it explicitly. There are two main approaches to introduce the integral, the first one by calculating the area under a curve and the second one by approximation. Accordingly, in the traditional teaching in German classrooms a geometric and an analytic definition are introduced to students. The former refers to the oriented area of a region under a curve and the latter to the limit of a sum of areas of rectangles.

In the German didactical literature, these different approaches have been intensively discussed. We, however, restrict ourselves to refer to some basic works in this area. Already in 1976, Kirsch called for an "intellectual honest" introduction of this concept in order to promote appropriate learning of students. Tietze, Klika and Wolpers (1997) differentiate two approaches that focus on the area aspect and the antiderivative while Blum und Törner (1983) refer to the mean value as a third approach. Hußmann (2002) points to the usefulness of the approximation aspect by applying the idea of cumulation, which offers the advantage of easily interpreting negative values.

Nevertheless, teaching integral calculus in German schools is characterized by stable patterns. According to Blum (2000), it is mainly oriented at schemata and formulae. Consequently students are lacking a conceptual understanding of the integral. He underlines this observation by referring to the following TIMSS task, which could only be solved by 23% of German secondary school students (see figure 2.5).

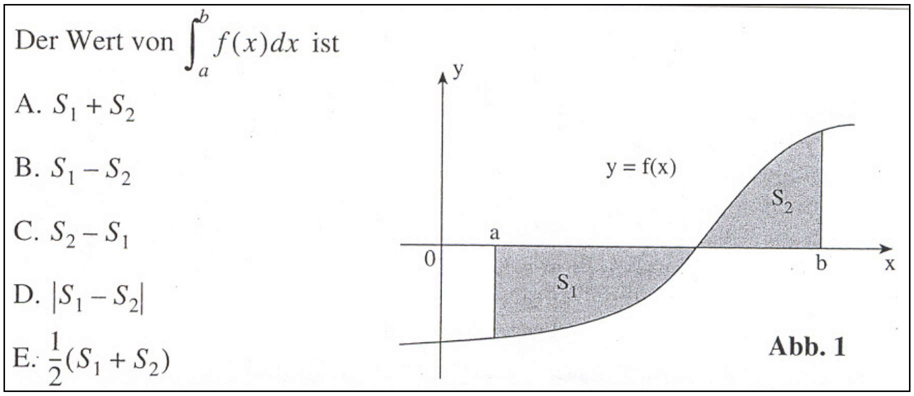


Figure 2.5. TIMSS task connecting area and integral (Blum, 2000).

The task shown in figure 2.5 considers the oriented area aspect of the integral and the results showed that many students just equalled the concepts of integral and area. This tendency also emerged in our study which will be outlined in the results section.

3. METHODOLOGY

The design of the study is conceptually oriented at the study of Rasslan and Tall (2002). The participants in our study were students from grade 12 of one German secondary school. The class consisted of 24 students, 14 female and 10 male students. The students were enrolled in a so-called *Leistungskurs*, an advanced mathematics course with five hours per week compared to three hours in the general course.

In this section, we present the guiding research questions and the design of the study including the problems that we employed.

3.1 Research questions

On the background of the aforementioned theoretical comments, we examined students' concept images and concept definitions of the integral. As mentioned before, our investigation was guided by a study from Rasslan and Tall (2002). Their underlying assumption is that letting students work on adequate problems provides insights into the concept images and concept definitions. The following research questions were central to our study:

- Which concept images and which concept definitions do students activate while working on problems around the integral?

- What alternative and individual concept images on the integral do students possess besides the intended ones?

- To what extent do the students reveal inconsistencies between the concept images and the concept definitions?

3.2 Design of the study

In order to get various information on the concept definition and especially the concept image of the integral, we employed different methods for data collection – called "triangulation" by Cohen, Manion and Morrison (2007). Likewise, Schoenfeld (2002) suggests employing multiple data sources in order to enhance the trustworthiness of the data.

Although our study is conceptually oriented at the one of Rasslan and Tall (2002), it is modified with regard to the German classroom and refined to get deeper insight into students' concept formation concerning the integral. In comparison to five tasks on the concept image, there is only one question on the concept definition in the study of Rasslan and Tall (2002), *In your opinion what is* $\int_a^b f(x)dx$ *(the definite integral of the function f in the interval [a,b])?* However, an analysis of German textbooks as well as lessons revealed that this question is too broad for the German situation (see also section 2.4). We therefore adopted a more sophisticated point of view and created three questions on the concept definition. Moreover, we explicitly asked the students not only to write down their solution but also to use sketches, to give explanations and to document their procedure. According to Rasslan and Tall (2002), we used five problems around the integral to get information about students' concept image. Furthermore, the concept image was investigated by inviting the students to draw a mind map. This is appropriated to represent in a graphic way the cognitive structure of a key topic including the related notions and the connections between them.

3.2.1 Mind Maps

Mind maps are considered as possibility to represent all aspects a person associates with a given concept. The name of the concept, "integral" in our case, serves as stimulus for recalling related contents and connections between them. These are organized in a diagram using lines to represent the relationships between different notions. A particularity of our task consists in the fact that the students were asked to enumerate the lines. This serves as information about the genesis of the mind map in the temporal order. Berger and Törner (2002) emphasize that this is a good means to get to know the central notions. In their opinion, the earlier a notion is recalled the more central it is to a person.

Beforehand, the students were shown via overhead projector an example of a mind map around the theme of function. Moreover, relevant aspects for the production of a mind map were recalled like those mentioned above.

3.2.2 Questionnaire

The questionnaire was employed in a pre-study, and a modified version was again implemented. Every time, the students had to work on the questionnaire in the classroom under supervision and were allowed to use a calculator.

Problems 1 to 3:

The concept definition was investigated by the following three problems:

Problem 1:

What do you understand by $\int_a^b f(x)dx$ *?*

Problem 2:
Give a geometric definition of the integral and an illustration.

Problem 3:
Give an analytic definition of the integral and an illustration.

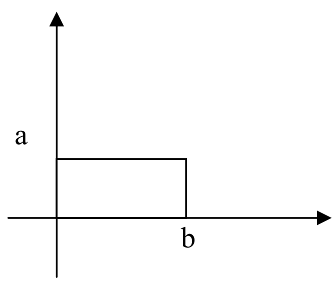
Problem 1 is meant to evoke students' associations with the symbolic notation of the integral. Problem 2 includes the area aspect while in problem 3, the approximation aspect is considered. With these three problems, we aimed at finding out which definition is familiar to the students and which aspects they remind in the respective context. As the students were asked to illustrate their solution by using sketches, the concrete conceptions on the integral can be checked. This includes, for example, the representation of the integral as positive or oriented area.

Problems 4 to 8:
The problems 4 to 8 are concerned with different aspects on the concept image of the integral.

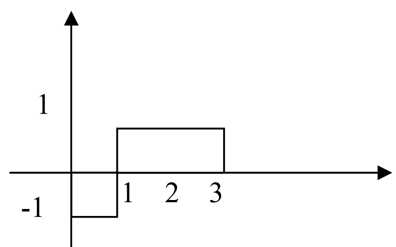
Problem 4:

Find a formula for the area by using integration.

a)

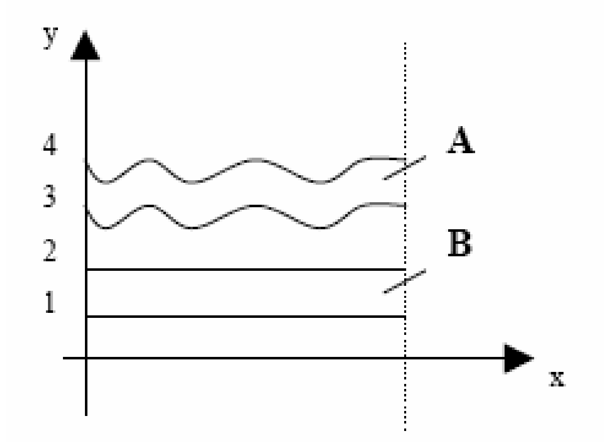


b)



Problem 4 was designed to uncover if the students were able to use the integral for the calculation of simple areas. We hereby examined the naturalness of the students to manage the symbolic notation of the integral, especially their capacity to determine the boundaries of the integral and the integrand. In the first part of the task, the indication of the integral for a rectangle located above the x-axis is required. In the second part of the task, pieces of the area are situated in the 4^{th} quadrant, that is, the integral here designates the oriented area.

Problem 5:
The picture shows two areas A and B. What do you think is correct for the relation between the areas?



- *The area of A is bigger than the one of B.*
- *The area of A is smaller than the one of B.*
- *Both areas are equal.*
- *Without any function given explicitly, it is not possible to answer this question.*

Problem 5 is concerned with intuitive approaches to the solution in contrast to more logically oriented considerations. Vinner (1997) pointed out that "the concept image can be considered as part of intuition" and that "in some cases the intuitive mode of thinking just misleads us" (p. 67). In addition, a similar task to problem 5 can be found in a study from Fischbein (1999) who emphasizes its intuitive character:

> We posed this problem to high school students. The immediate reaction of these students was that the two areas are not equal and this means that there is nothing to prove. This was the *intuitive*, direct, apparently self-evident reaction to our problem. In reality, the two areas *are* equivalent, but this may be proven only indirectly by a logical analysis. (p. 20)

Problem 6:

a) Find the area bounded between the function f(x)=sinx and the x-axis over [π,2π].

b) Calculate the integral: $\int_{-\pi}^{2\pi} \sin x dx$.

In problem 6, conceptions on the area aspect of the integral are raised. Part a) asks for the calculation of the area that includes the graph of the sine function with the x-axis in the interval [π,2π]. In part b), the calculation of the integral of the sine function in the interval [-π,2π] is required. If the two tasks are considered in a holistic way, the illustration of the position of the area will facilitate the solution. A sketch, for example, would enable the students to answer problem 6b) immediately – assumed that they possess a profound understanding of the integral.

Problem 7:

How would you proceed to calculate the integral $\int_{-1}^{1} \sin(2x^3)dx$ *?*

In problem 7, the competent handling of the integral by initially using visualization is tested. As the function is odd and point symmetric, its integral on the given interval [-1,1] equals zero (see figure 3.1).

Compared to substitution or integration by parts, these simple and elementary considerations save time and laborious calculations.

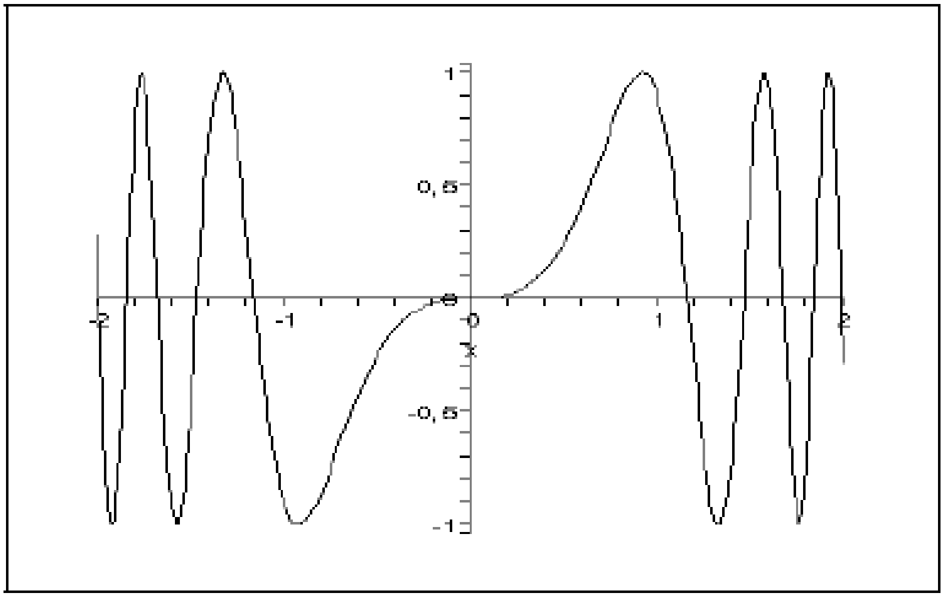


Figure 3.1 The graph of the function $f(x) = \sin 2x^3$.

Problem 8:

Give an example for a non-integrable function and explain your choice.

In problem 8, the students are asked to give an example for a function that is non-integrable and to explain their example. The underlying assumption is that the use of examples and counterexamples contributes considerably to the deep learning of concepts (Vollrath, 2001) and we wanted to know if counterexamples are also part of students' concept images.

4. RESULTS AND DISCUSSION

In this section, we present the results of our study. We first report on the problems that we designed to prompt the concept definition. However, if there become evident interesting aspects related to the concept image, we also mention these. We then present the results on the tasks explicitly created to evoke the concept image. Although we also used mind maps to investigate students' concept images we do not present these results in this paper and we therefore refer to Rösken (2004). To support our interpretations, we insert excerpts of students' answers.

4.1 Concept definition

Problem 1:

All students worked on the first question. The responses reveal two main aspects. On the one hand, the students explain the symbol of the integral; on the other hand, they refer to the area aspect. Table 4.1 shows the distribution of the answers.

Table 4.1. Students' answers to problem 1.

Students' answers	**Number**
The integral from a to b of the function f.	12
The area that the graph of f includes with the x-axis in the intervall [a,b].	12
Others	3

The sum of the students in table 4.1 does not equal 24 because three students mentioned both aspects. The twelve students that relate to the symbol of the integral the area aspect use quite vague formulations to express their ideas. The geometric conception associated with the integral is linked to simple area aspects. The following quotes from students' answers serve as examples to illustrate this:

- With this formula, you can calculate a certain area which is limited by the boundaries a and b.
- This is the area that the graph f(x) includes between the interval [a,b] and the x-axis.
- The graph f(x) includes between x=a and x=b an area in relation to the x-axis which is determined through the integral.

The notion of oriented area is not mentioned by any student. Three students are not able to give an appropriate answer:

- It is the average value of f(x) related to the domain from a to b.
- Hereunder, I understand the function f(x) that describes the graph in the interval from a to b in relation to the variable x.
- This task deals with a function f(x) that is located in the domain from a to b.

It is worth mentioning that in the students' answers the explication of the symbol of integral refers more to the concept definition while the interpretation as area is more related to the concept image. Finally, the results from Rasslan and Tall (2002) have already shown that this question is only in a restricted way appropriate to reveal representations on the concept definition.

Problem 2:

The analysis of the answers to this question is also guided by the formation of categories. The geometric definition of the integral relates to the geometric notion of area. As already mentioned, in German textbooks, the integral is interpreted as oriented area. The students' answers are put into categories in order to structure them according to their quality. First, we distinguished between three main categories. Within these, we further differentiate, as can be seen in table 4.2:

Table 4.2. Students' answers to problem 2.

Category	Content	Number
G1	**Appropriate definition**	**16**
G1a	Oriented area	1
G1b	Area for functions situated above the x-axis	15
G2	**Inappropriate definition**	**6**
G2a	Incorrect or incomplete definition	4
G2b	Only visual representation	2
G3	**No answer**	**2**

In order to clarify these categories, we further explain them and give examples from students' answers. In category G1a, we put the answers that refer to the formal concept definition. An important criterion of distinction to category G1b is the notion of oriented area. Figure 4.3 is an example to illustrate category G1a.

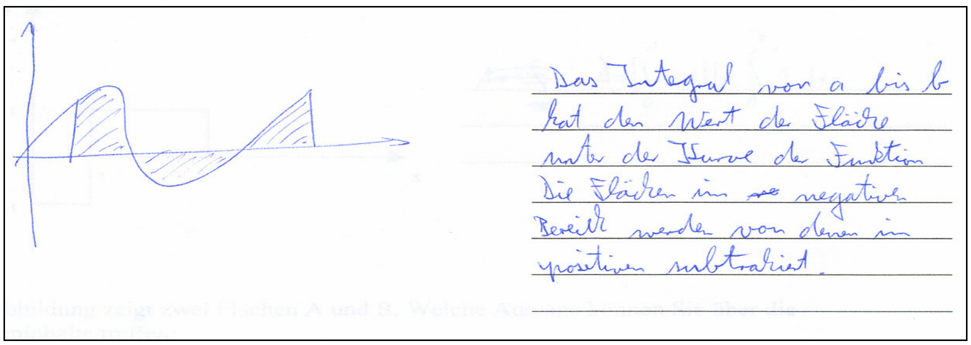


Figure 4.3. Example of a student's answer categorized as G1a. [Translation: The integral from a to b has the value of the area under the curve of the function. The negative located areas will be subtracted from the positive ones.]

In category G1b, you can find the definitions that relate to an area of a function that is situated above the x-axis. Figure 4.4 serves as illustration of this category:

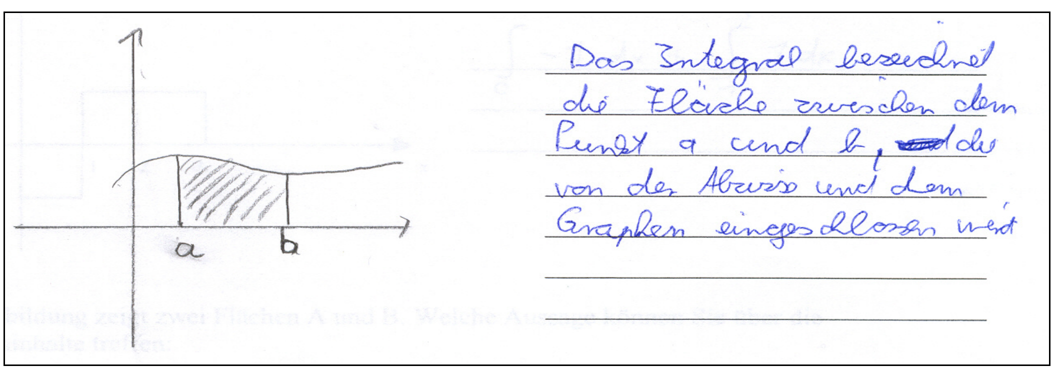


Figure 4.4. Example of a student's answer categorized as G1b. [Translation: The integral indicates the area between the points a and b, which is included by the x-axis and the graph.]

In the two classrooms in which we employed this questionnaire, the introduction of the integral calculus was based on representations like the one illustrated in figure 4.4 in order to motivate the area aspect. However, given the limited understanding that is expressed in answers like the one in figure 4.4, the question arises whether these students have developed an adequate understanding of the integral. In the questionnaire, there are two problems that are concerned wit the interpretation of the integral as sum of positive and negative areas. The interesting question is, *How are students that possess a concept image in the sense of figure 4.4 going to solve theses problems?*

Altogether, students' answers to problem 2 show that the area aspect is strongly linked to the image of a certain function, like the one sketched in figure 4.4. This conception proves to be rather demonstrative and is remembered by the majority of the students.

Problem 3:

In the analytic definition, the area under a curve is approximated by polygons. This definition was initially introduced in the classroom for non-negative, continuous functions. Only later, this was concretized for the sum of positive and negative areas. For the interpretation of students' answers, we again distinguish between three main categories. The differentiation within these is more fine-grained than in question 2. This is due to the complexity of the analytic definition in comparison to the geometric one. Table 4.5 illustrates the categories and subcategories:

Table 4.5. Students' answers to problem 3.

Category	Content	Number
A1	**Appropriate definition**	**6**
A2	**Inappropriate definition**	**18**
A2a	Good definitional attempts	9
A2b	Vague or absurd definitional attempts	8
A2c	No definition, only visual representation	1
A3	**No answer**	**6**

In nine students' answers, good definitional attempts become obvious. These students do know the constitutive aspects of the definition. However, it is difficult for them to express the connections appropriately. They describe the process of approximation but, at large, they are not able to define it adequately. A comparison between problem 2 to 3 shows that it is much easier for the students to define the integral using the geometric aspect. Yet, one has to take into account that this definition is restricted to the image of a specific function. The approximation by polygons turns out to be demonstrative as well and is used in 15 definitional attempts. However, the main ideas cannot be expressed as easily as in the case of the geometric definition of the integral.

Problem 4a:

Half of the students are able to give the correct result as shown in table 4.6. More interesting and insightful is analyzing the incorrect answers in detail.

Table 4.6. Students' answers to problem 4a.

Category	Content	Number
1	Correct result	12
2	Difficulties to name the integral	9
3	Calculation of the area without using the integral	3

Among the incorrect answers in category 2, the following terms can be found:

- $\int_a^b kdx, \int_a^b adx, \int_a^b a(x)dx, \int_a^b f(a)dx, \int_0^b ada.$

One difficulty for the students is to name the limits of integration. It is evident that finding the integral for the given image conflicts with the standard notation. Furthermore, the students have major problems to recognize the given constant function as a possible integrand. Obviously, they are missing an x-term.

Figure 4.7 shows a solution that two students gave in the pre-study. Even if this was in the pre-study, we consider their procedure as worth mentioning here as well. These students solved the conflict mentioned above by drawing a supporting straight line as shown in figure 4.7. Hence, they obtained the answer to this problem in a creative though complicated way.

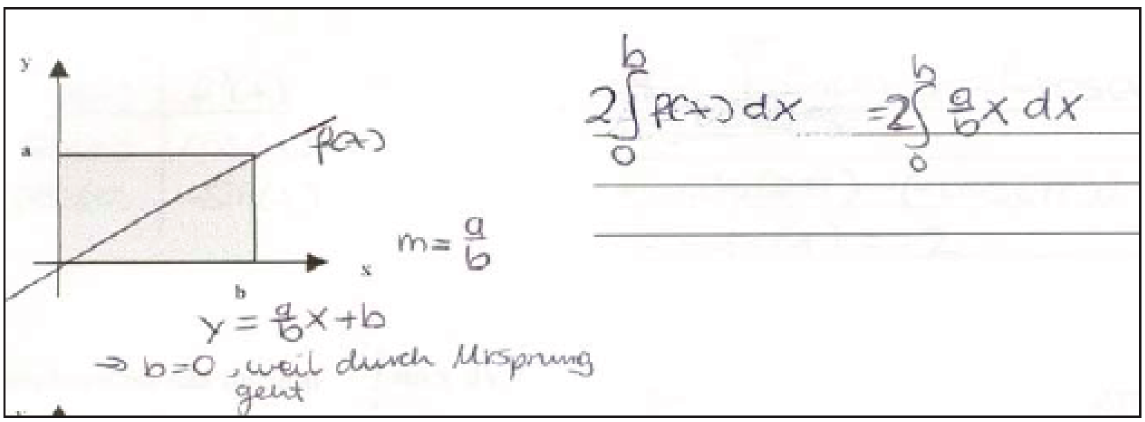


Figure 4.7. One student's solution to problem 4a.

This task documents that many students associate a certain image of function to the integral. In the case of the two students, they try to produce this concept image in order to be able to apply their knowledge.

Problem 4b:

As can be seen in table 4.8, ten students gave the correct solution and 14 an incorrect one. Again, we have a closer look at the incorrect solutions:

Table 4.8. Students' answers to problem 4b.

Cat.	Students' answers	Number	Aspect of the concept image
1	$\int_0^1 -1dx + \int_1^3 1dx$, the orientation of the area is not considered	4	Elementary conceptions of area
2	$\left\lvert \int_0^1 f(x)dx \right\rvert + \int_1^3 f(x)dx$, likewise a(x), f(1) or f(-1) are named as integrands	7	Strong attachment to the traditional notion of integral
3	Elementary calculation of the area	2	-
4	$\int_0^3 (x-1)dx$, calculation of the area via a straight line	1	Calculation of the area is associated with a certain image of function

The answers show that the respective concept image of the students is linked to certain conceptions that constitute an obstacle for working on the task. In category 1, the students do not consider the position of the area while calculating it using the integral. Their concept image is bound to simple conceptions of area. In category 2, the students perfectly take into account the position of the areas, the mistake is due to another reason: they have difficulties to name the integrand. Their concept image for the symbol of the integral is associated to a certain type of function. In order to keep in line with this conception, these students indicate as integrand f(x), a(x) or f(1) and f(-1). One student gives the right solution but evaluates her result by the following statement:

- $\int_0^3 1dx$: Not possible, because this is a constant function and there is no x in it and that's why it is not possible to put in the limits.

It becomes evident here that the concept image does not exist in a consistent way. First, the student gives the correct answer but then notices that it is not in line with other parts of her conceptions. The conflict is solved by preferring the part of the concept image that is related to a specific function in the notion of the integral.

The students that we put in category 3 do not pay attention to the task and calculate the area without using the integral. They draw on elementary ways to calculate the area.

In category 4, the student tries to calculate the area by considering a "supporting straight line". He obtains the function f(x)=x-1. Again, it becomes apparent that the concept image is associated to a certain image of function. The area under a constant function does not seem to correspond with this conception.

The two main categories, namely 1 and 2, do not appear jointly. This means that students who were put in category 1 indicate the right functions while students that were put in category 2 have difficulties to name the integrand but possess correct conceptions on the area. One student combines the elementary calculation of the area with the integral calculus by writing the following:

- $\int_0^1 a^2 dx + \int_1^3 abdx$.

Summarized, while the difficulties to find the integrand remained, the problem to name the limits of integration minimized due to the concrete numbers provided in the illustration. However, a new obstacle emerged because of the orientation of the areas. Instead of the area, the students calculated the integral. Some students solved this conflict by shifting the square above the x-axis (see figure 4.9).

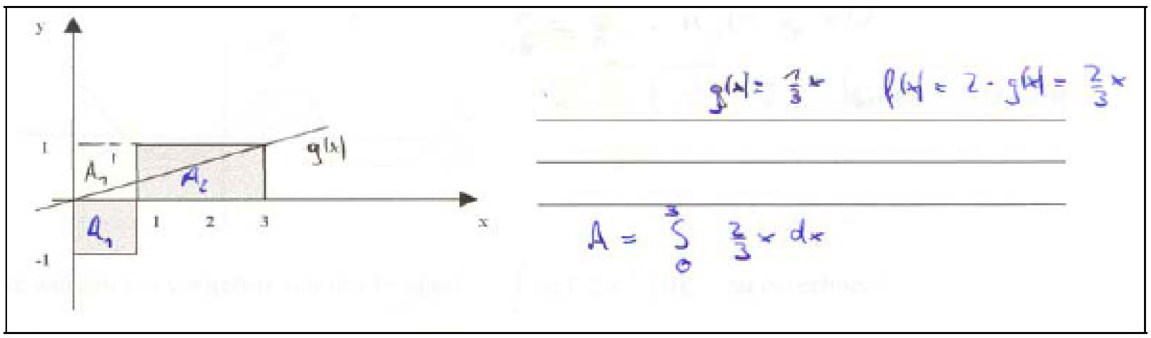


Figure 4.9. One example for students' solution to problem 4b.

Problem 5:

The areas A and B are equal. There are different possibilities to compare the two areas. 20 students give the right solution. Of these students, 17 explain this by stating that the respective functions have the same distance on the whole interval. Exemplarily, the following two answers document this:

- The areas are equal because the distances of the two limiting lines are equal.
- The x-section is the same, the y-section (that is between the limiting lines) as well, hence the areas are identical (for every x the difference of the limiting functions is identical).

One student refers to Cavalieri's principle. Two students draw an additional line and use it to show the equality of the areas. Figure 4.10 shows this solution.

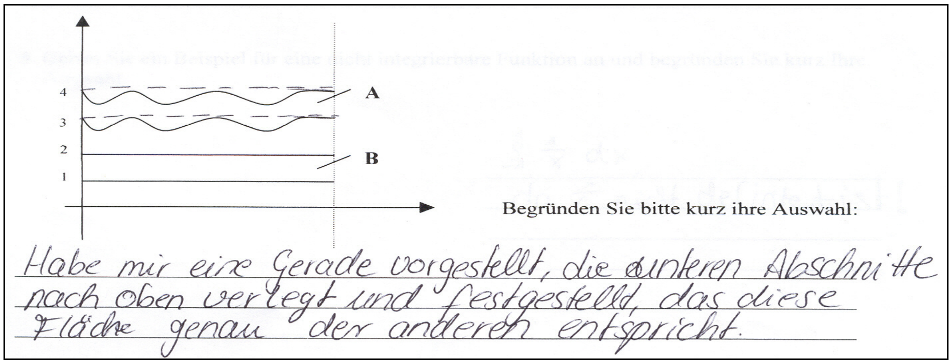


Figure 4.10. One student's solution to problem 5. [Translation: I imagined a straight line, put the sections from below to above and noticed that this area just fits the other one.]

Two students argue that the area of A is bigger than the one of B. They give the following explanations:

- If I stretched out the area A, I would have two horizontal lines that would be longer than the lines of area B.
- As the two functions that include A, exactly as for B, always have the distance 1 but are waved and have the same limit, A > B.

Two students claim that an answer to this question is not possible because there is no function indicated. They state that – only by referring to the illustration – you cannot assume the parallelism of the functions that limit A. A student in the pre-study gives another interesting answer:

- Because of the curve respectively the waves, the graphs of A need as much as those of B. However, the graphs of A are much longer, hence, the area is also bigger than B. Example: The intestine in the body is much longer as one imagines but as it is curved and folded, it fits in our body.

It becomes obvious that the concept image does not only contain mathematical elements. The process of solving problems also involves everyday life experiences and considerations. All in all, the results show the importance of already existing knowledge and intuitive conceptions for the students' concept images. The concept images build on the experiences and knowledge that also might stem from other contexts. Spontaneous and intuitive considerations are assigned priority while analytical verifications are neglected.

Problem 6a:

The area that the students were to calculate is situated in the 4^{th} quadrant. The area that the graph of f(x) includes with the x-axis in the interval $[\pi,2\pi]$ equals 2. Ten of the 24 students are able to give the correct solution. The 14 incorrect solutions are described and analyzed in table 4.11:

Table 4.11. Students' answers to problem 6a.

Cat.	Students' answers	Number	Aspects of the concept image
1	-2 as result, i.e., negative area	7	integral = area
2	Incorrect antiderivative (cos x, - sin x)	5	-
3	Other arithmetic errors	2	-

Seven students calculate the area instead of the integral. Their answer results in giving a negative value as area of the function. On the one hand, these students do not use visualization to approach the problem. On the other hand, they do not scrutinize the negative value of their result. Some even add [FE] in order to indicate that they refer to the size of an area. In their concept image, the calculation of the integral is either equal to the one of the area or they do not consider it necessary to take into account the course of the graph. No student at all sketches the graph of the function.

Five of the seven students in categories 2 and 3 use the absolute value for the integral and document at least a correct procedure for the area aspect.

Problem 6b:

Considering the solution to problem 6a as well as the course of the graph, the value of the integral can immediately be given, namely – 2. Here, nine students answer the problem correctly and 15 incorrectly. The analysis of the incorrect answers is similar to the one in problem 6a. This is why we abstain from presenting them here again. However, it is remarkable that some students continue to calculate the area instead of the integral as required in 6a. They give as answers a positive value, some of them even mention explicitly A=2.

This task aimed at broaching the relationship between the integral and the related area. The results show that many students are not able to clearly distinguish between these two concepts. Their concept images equate the integral with the area. This relationship exists in both directions, as the two parts of the problem show.

Problem 7:

The integral equals zero. Eight students propose to work out the integral by finding the antiderivative, nine students by substitution and three students by integration by parts. Two students do not answer this question at all. Only two students take into account the course of the given function:

- Integration by substitution, using opposite boundaries of integration.
- The function is point symmetric to the point of origin; hence the integral (not the area) equals 0.

The first answer contains a vague allusion to the boundaries of integration. In the second answer, the student elaborates on this idea even further.

Already in the previous task, the sine function was subject of the considerations. Here, it is combined with another odd function. Therefore, the point symmetry of the graph is still obtained. The solutions to this task show an explicit bias towards an algorithmic approach even though the visual one would have been significantly easier. Aspects related to the calculation like, *How can I calculate the integral?*, come to the fore in students' considerations while relevant aspects of the integral take a back seat. They are captured in the procedure of calculation and fixed on algorithms instead of using the concept in a creative and flexible way. Interestingly, the mental images and conceptions that form the concept image as a whole are eclipsed. The introduction and the development of the integral are highly based on visual representations. However, these visual aspects of the integral do not seem to have the same relevance for the students while working on concrete problems. The starting point for students' reflections is not an appropriate visualization but rather oriented at the choice of an appropriate procedure.

Problem 8:

In this question, students were asked to indicate counterexamples that they knew or were able to create on the basis of their intuitive conceptions. Seven students mention as example of a non-integrable function a function with a pole:

- Spontaneously, only functions that have poles like the tangent function cross my mind. Those can be handled by considering limits.

In this context, six students refer to the function f(x)= $\frac{1}{x}$. The following statements can be found in the students' answers:

- Functions with poles: with poles, you cannot integrate $\int_{-1}^{1} \frac{1}{x} dx$.
- $\int_{-1}^{1} \frac{1}{x} dx$ because $\frac{1}{0}$ is not defined.
- $\int_{0}^{2} \frac{1}{x} dx$. This is non-integrable because 0 is not allowed for x, there is a pole.

These students justify their examples by pointing to the existence of a gap in the domain. Three students mention functions whose domain is the whole number line:

- x = IR (for example, y = 1, y = 2, y = 3, y = 1.0006.783.986). These functions are not integrable because in an arbitrary interval, they do not include any area.
- $\int f(x)dx$ because no intervals are given.
- f(x) = 0. As this function possess no area, it is not possible to integrate.

One female student connects the integrability with the differentiability:

- f(x) = sign(x) non-differentiable → non-integrable.

Another interesting answer is given by a female student:

- $\int_a^b 5x \cdot 10x^2 \cdot 20x^3 \cdot 40x^4 dx$. We have only learned the product rule for two factors. Maybe there is a rule that allows for integrating this function as well but I do not know it.

The student connects the integrability with the existence of an antiderivative and mixes two different aspects. This idea is also hinted at in the following answer:

- I do not know it exactly, many [functions] come to my mind but I do not know if somebody could not integrate them in another way.

Only one student gives as an example the Dirichlet function.

The answers show that the concept image of the students contains various conceptions and explications to the notion of non-integrability that do not always describe the circumstances adequately.

5. CONCLUSION AND EDUCATIONAL IMPLICATIONS

The model of concept image and concept definition allows for analyzing the cognitive processes influencing the learning of mathematics by simultaneously considering mathematical characteristics. In general, understanding new information is essential in order to integrate new aspects into previously existing knowledge structures. Obviously, in learning mathematics students are to build rich and comprehensive concept images related to a specific notion. These concept images include own associations which make it possible for the students to grasp the ideas behind a mathematical notion. However, the formation of a concept image is an ambivalent process, in the sense that important aspects of the formal definitions are often not adequately represented. As a result, difficulties in students' learning might occur. The concept images could include critical aspects in comparison to the concept definitions.

In our empirical study we analyzed students' concept definitions and concept images of the definite integral as well as inconsistencies between them. In what follows, we sum up some of the main information we got out of the results to the problems. Our findings indicate that the students have developed rich concept images. The results also showed that most students knew the relevant aspects around the concept. As expected, the definitions were rather weakly represented, the findings were even worse for the analytic one. While working on the tasks to the integral many students ran into serious problems. The results raised the following conflicting issues of students' concept images:

- The concepts of area and integral are not distinguished.
- The symbol of integral is connected to a specific type of function.
- The calculation of the area is restricted to a specific graph of function.

It becomes apparent that the concept image has the character of a schema, including different ideas connected to the integral. If a task conflicts with the concept image, assimilation processes will occur. Parts of the task were ignored or interpreted in a different way. Further, we could see that the ideas of students cannot only be explained by their current concept image but ideas on other mathematical objects, too. We therefore conclude that a concept image cannot be seen as a single unit but is integrated in a networking of different concept images with manifold relations between them (see figure 5.1).

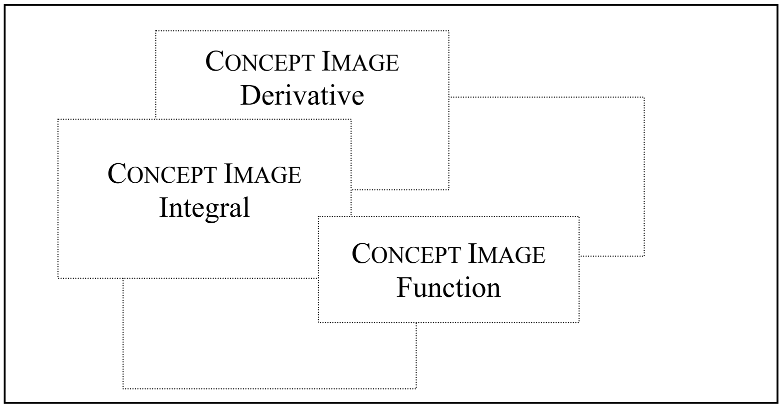


Figure 5.1. Networking of concept images related to different mathematical notions.

This networking aspect became obvious because of the difficulties students encountered when working on problem 4. Particularly, their evoked concept image of a function affected the problem solving process. That is, the flexibility in dealing with the current concept is restricted by difficulties related to other topics. The integration of knowledge is a sensitive process, problems in application situations easily occur.

Intuitive ideas are an important and decisive part of a concept image, as could be seen in the results to problem 5. Not only mathematical knowledge but also other experiences have a considerable effect on problem solving. Intuitive associations occur spontaneously, an analytic verification is often neglected. In most cases, these intuitive ideas dominate the concept image and often contain critical aspects compared to the concept definition.

In problem 6, the connection between the idea of an integral and the idea of an area was raised. As already could be concluded from students' answers to the geometric definition, the relation between the two notions is restricted to just one aspect. When working on the problems, these misconceptions became even more apparent. Moreover, some students are guided by the principle of always employing absolute value when calculating an area by using integration. For many students this rule is part of their concept image, and they are therefore not able to clearly distinguish between the concepts of area and integral. Furthermore, students did not consider it necessary to think about the course of the graph.

Another interesting aspect emerged from the results to problem 7. The evoked concept image of students was primarily restricted to techniques and procedures to the disadvantage of visualization, for example. That is, more creative approaches are not applied although they would be helpful. This focus on algorithms and calculations can also be seen in the other problems. We concentrated on this aspect in another paper in more detail (Rösken & Rolka, 2006). When we asked students for a non-integrable function in problem 8, a lot of connections to other mathematical topics occurred. What became apparent was that students had problems to link ideas from different contexts and to bring together different concept images. Without doubt, students possess a lot of conceptual knowledge as could be seen in the results.

Concerning the methodological approach, we conclude that we got wide-ranging information on students' concept images and concept definitions due to applying different methods. This study is part of ongoing research and in a follow-up study we are going to improve the following issues. First, we will additionally conduct interviews with the students in order to get deeper insight into their approaches, difficulties and subjective concept formations. Second, we will discuss some of the eye-catching students' results with the teachers to sensitize them for their students' intuitive approaches.

Finally, we can say that Tall and Vinner (1981) provide a powerful model to deeper analyze the construction of mathematical knowledge. Related to classroom praxis, the following questions can be raised, *How can students be inspired to build rich and useful concept images? What should teachers know about students' corresponding cognitive processes?* The critical aspects of the concept image that have been described in this paper could also offer a possibility to make the underlying conflict transparent for both students and teachers. Essential is to help students integrating intuition but unfortunately, many teachers are – for different and understandable reasons – primarily concerned with judging and evaluating "incorrect" solutions. These are rashly labelled as misconceptions instead of using their inherent potential for teaching.

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