FROM MORPHISMS TO MATRICES

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Abstract: *The "New Math" was an attempt to implant the spirit of Bourbaki in the classroom. In linear algebra this meant introducing coordinate-free linear mappings. However, all concrete examples involved matrices in low dimensions with few convincing applications. Forced by the dissonance between the students' expectations and his own, the author finally designed a course in matrix algebra, whose beginning and end are sketched in this article. The last section gives an elementary proof of an old theorem of Camille Jordan.*

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1. Introduction

Clashing Beliefs: In my three decades as a university professor, it frequently happened that I was assigned an introductory course in Linear Algebra. My first attempt at this, though filled to the rim with pedagogical and mathematical enthusiasm, was an unmitigated disaster. In 1967, I had just come back to Canada after ten years spent as a graduate student in the United States and post-doc in Germany, with all the innocent beliefs grown and fostered in those blessed situations. For present purposes, they can be summarized in two points: (i) Though in perpetual evolution, mathematics provides a peculiarly firm kind of knowledge which is not available in other ways; it is worth knowing "*pour l'honneur del'esprit humain*"(Dieudonné,1987), if for no other reason.(ii)The role of the mathematics teacher was to draw the student's mind into this mighty river by finding suitable entry points (exercises, examples, puzzles) and describing its sweep and flow with the greatest possible clarity. Dream on, eh? In those intervening years, I had not been insulated from teaching, but events had conspired to provide me with (small classes of) students who were mathematics enthusiasts with belief systems similar to mine. My rude awakening, which began when Günter had just turned 20, came through the encounter with larger groups

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of young people who marched to a very different drummer. As I grew to know them, in and after class, during office hours and at parties, they told me how *they* saw our subject:

(a) Though hard to understand, mathematics was an essential tool in every theoretical and practical science (hence also for academic advancement), wherefore its many facts and procedures had to be memorised, ideally forever, but in practice till the next exam.

(b) The role of the mathematics teacher was to show you, without irrelevant frills, how to handle this tool , ideally for various complex situations, but in practice for the exercises in the textbook and (especially) potential exam questions.

Such a divergence of perspective was bound to inhibit communication and had to be reduced. But how? The students were caught in a system which proclaimed Academic Freedom at convocation time, but otherwise practised paternal despotism by *prescribing* mathematics (among other things) and then certifying them as winners or losers via questionable "evaluation" techniques. To get into medicine in Canada, you needed Calculus (in Germany, you needed Latin), yet nobody would supply a reason I could understand. My brother-in-law, a neurosurgeon, bitterly challenged me years later to justify it. Could I have answered that it was "pour l'honneur de l'esprit humain"?

Whether or not they believed in the benefits of memorisation (which, by the way, seem to be non-trivial), the students' beliefs (a) and (b) can be seen as a rational response to the demands of the Kafkaesque machine that processed them. But where was *my* rationale? While there were many ways of adapting item (ii) to the realities of the day without undue corruption, I could find no honest way past the bottom line of (i): there are only two ways of knowing, namely observation and deduction, of which the second is uniquely human and most intensely cultivated in mathematics.

Since it has become fashionable to quibble with this kind of statement, I need to clarify my modest version of it, before getting back to the gap between my beliefs and the students'. Although I am strongly sympathetic to the views expressed, for instance, by Alain Connes in his famous dialogue with Jean-Pierre Changeux (Changeux and Connes, 1995), I am not defending any ideological position. I just find that there is something delightfully sturdy about well-weathered mathematics, irrespective of whether you think it is discovered or invented.

To mention one tiny example: there are exactly five regular polyhedra, having 4, 6, 8, 12, 20 faces respectively, and the dimensions of the last two are tied up with the Golden Mean. Such facts could not be changed even by the Divine Will, which is why they were so unpopular with medieval theologians. Yet they are visible to you and me in the darkest night, and any questioning of their validity only strengthens them.

Nature Versus Texture. This robust nature of mathematical truth has been a constant in the over two thousand years since Archimedes (at least), and so has the basic method of pursuing it: reasoning and imagining, staring at sand-trays, black-boards, or ceilings, lost in thought. What *has* changed, however, is what is seen behind those ceilings. It is more than just style, as it really produces mental imagery of a different sort. For an easy example, take "Wilson's Theorem": every prime p divides the number $(p-1)!+1$. The present way of explaining it is to say that, in the field of p elements, the non-zero members other than 1 and -1 come in reciprocal pairs, which are knocked out in the product over all of them, leaving -1 as sole survivor. The point is that numbers now stay in the back of the mind as fleeting shadows.

In linear algebra (and analysis), many of the *dramatis personæ* of the nineteenth century, in as much as they already existed, were similarly reduced to faceless "vectors", denizens of abstract "vector spaces" tossed about by "linear mappings", a.k.a. transformations or operators, and these were later seen as special cases of "morphisms". Without this radical pruning of formerly obstructing trees, which soon gripped all of mathematics, the gigantic forests explored in the twentieth century would never even have been seen.

By the middle of the twentieth century, this recasting of mathematics, which had begun fifty years earlier, had given it a different texture or "feel", and become the dominant way of thinking about it. Through the naïveté of most mathematicians and the wrong-headed ambitions of a few trail-blazers, the educational establishment came to believe that this way of thinking could begin in grade school. My misadventures with linear algebra (and I was not alone in this leaky boat) suggested that it was not even suitable for the college level.

Getting back to beliefs, this is not the place to describe all the little compromises in the students' position (a) and my position (ii) which made life more harmonious: they are easy to imagine and not particularly original. The hard core of disagreement was over the nature of mathematics in (i) and the *real* purpose of taking the course in (b). It gradually dawned on me that, to the students, the course content was not defined by the syllabus but by the questions on old exams. So, I went and studied them.

They had a strong family resemblence. There was always one "big" (say, 4×5) system of linear equations, whose homogeneous and inhomogeneous solutions were highly endowed with points, with a subsidiary quickie on such matters as rank and nullity, starved for points. There was also one "theoretical" question about kernel and image of a linear map, extending the basis of a vector-space and the like, also lightly weighted. The rest was about matrices and coordinatised vectors in two or three dimensions, weighted at least 3:2 in favour of the lower dimension.

Compared to Calculus, linear algebra had at least two extra hurdles for the learner (engineering students called it "Mystery Math"): it lacked a central theme, like the connection between derivatives and integrals, and it always vacillated between the parallel worlds of vector-spaces and matrices, making it difficult for mental images to jell. In reorganising the course to fit the exam, writing my own slim set of notes with exercises and all, I came down on the side of matrices and chose "standard forms" as the central theme.

Spending one third of the course on 2×2 matrices, allows students to become thoroughly familiar of working with matrix phenomena which also occur in higher dimensions. In particular, eigenvectors quickly show up in the kind of "phase-plane" diagrams you see in books on differential equations (cf. V.I. Arnold, 1980, p.26 ff). After all, when you solve the latter by Euler's Method, they turn into difference equations, whose solution orbits are of the form $\{A^n X \mid n = 0, 1, 2, \ldots\}$, where A is a matrix (in the linear case). The programming needed to make these visible on a computer screen is trivial, and playing with the program is fascinating. The question of eigenvalues comes up naturally, and is easily handled by quadratic equations.

Instead of such diagrams, this article includes two stories (*Ahorita Vengo* and *I conigli di Leonardo*) on this theme. It then shows, in a somewhat compressed manner, how the necessary algebra was delivered. Of course, everything gets more difficult with the step up to three dimensions, but the basic drift can be maintained. The final chapter, which I got to only once, with a particulary bright class, deals with standard forms for $n \times n$ matrices. An elementary rendition of the Jordan form ends the paper.

2. Preliminaries.

A *matrix* is just a rectangular array of numbers, in itself nothing mysterious. However, the algebraic interactions of such arrays, and their geometric manifestations, open up a world of fascinating mathematical phenomena. To keep things simple, we'll initially limit ourselves by and large to just two kinds of matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

When we say "matrix", we shall usually mean the first of these types, preferring the term "vector" for the second.

Vector quantities abound. For instance, instead of considering an investor's assets as a single amount, you might break it down into separate values for stocks and for bonds; or, instead of looking at a total population of eagles, you might count adults and juveniles as separate subpopulations. To study how these numbers change from one time period to the next, you generally have to consider four coefficients: the growth rates of x and y

by themselves, as well as the factors by which they hinder or enhance each other. This is where matrices come in; most frequently they occur as *multipliers of vector quantities*. The multiplication of the vector X by the matrix A shown above is defined as follows:

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}.$$

If the terms b and c are both $= 0$, the matrix A is called *diagonal*. In that case, the components of AX are just ax and dy with no cross-terms, and the problem does not really need vectors; it is just one ordinary 1-dimensional problem on top of another. One of our major themes is the search for ways of simulating this happy state of affairs ("diagonalization") or at least to come as close to it as possible.

Before we can go anywhere on this meager lead, we need a minimum of information on matrix algebra. Of course, matrices can be added together and scaled in the most obvious manner (entry-by-entry), but multiplication should respect the action on columns ("vectors") given above:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & u \\ y & v \end{bmatrix} = \begin{bmatrix} ax+by & au+bv \\ cx+dy & cu+dv \end{bmatrix}.$$

If we think of B as two columns side by side, say $B = [X, V]$, multiplication by A amounts to $AB = [AX, AV]$. It is amazing that this makes 2×2 matrices into a viable algebraic system. The interplay of rows and columns is necesary for the "associative" property $A(BC) = (AB)C$, but that is not as obvious as one might wish.

On the other hand, a careful look at the definition reveals that the two distributive laws, namely, $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$, do in fact hold. However, we shall presently find examples to show that $AB \neq BA$ in general, and that $AB = 0$ is possible even if $A \neq 0$ and $B \neq 0$. Such examples come up already if one of the factors is a *diagonal* matrix $D(\alpha, \beta)$:

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \beta c & \beta d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} \alpha a & \beta b \\ \alpha c & \beta d \end{bmatrix}.$$

Note: left multiplication by a diagonal scales the rows, while right multiplication by a diagonal scales the columns of the other matrix. If $\alpha = \beta = 1$, we have the *identity matrix* $I = D(1,1)$ which, as a multiplier, has no effect at all: $IA = AI = A$ for any A.

Before doing any more symbolic work, let us look at two illustrations.

Ahorita Vengo. This is the story of a ficticious Mexican capitalist, a lady named Ahorita Vengo, who holds considerable sums of both Canadian and American dollars, but must do all her calculations and transactions in Mexican pesos. If x and y denote the number of millions of pesos invested in Canadian and American currencies, respectively, the annual growth of these monies can be represented by the matrix multiplication

$$A \begin{bmatrix} x \\ \\ y \end{bmatrix} = \begin{bmatrix} 1.09 & -.04 \\ \\ -.03 & 1.05 \end{bmatrix} \begin{bmatrix} x \\ \\ y \end{bmatrix} = \begin{bmatrix} 1.09x - .04y \\ \\ -.03x + 1.05y \end{bmatrix}.$$

For now, you may wish to skip the derivation of this equation, but if you are curious, here is how it works: firstly, there is a higher annual interest rate in Canada (12%) than in the US (9%); secondly, the Mexican government taxes Canadian holdings at a lesser yearly rate (6%) than the American ones (8%); thirdly, these taxes on North American accounts are payable half in Canadian dollars and half in US dollars, irrespective of their source.

Ahorita is annoyed by the irregularity of this growth pattern. If she puts 3 million pesos into Canadian ($x = 3$) and an equal amount into American ($y = 3$), she winds up, after one year, with 3.15 in one currency and 3.06 in the the other. Unfortunately this does not mean that x consistently grows at 5% and y at 2%. If she had split her 6 million by putting $x = 4$ and $y = 2$, the results would have been 4.26 and 1.98, respectively, suggesting growth rates of 6.5% and -1%. Not even the totals are the same: the 6 would have grown to 6.21 under the first scheme and to 6.24 under the second.

Her sister-in-law, Megusta Poco, is a professional broker and tells her about the magic $40 - 60$ split, suggesting that she invest 2 million in Canadian and 3 million in US money. Sure enough, if she puts $x = 2$ and $y = 3$, she gets 2.06 and 3.09 for the next year, a clean 3% gain across the board. Best of all, the $2 : 3$ ratio is maintained, and the same percentage increase will be repeated every year, great! She leaves the 5 million in Poco's care and turns her attention to personal matters.

She still has 1 million left to play with, so a few days later she goes to consult Poco's arch-rival Loquiero Mucho, a somewhat shady dude, feared and respected for his uncanny knack of making money. Mucho says that there is another magic ratio, and on receipt of a hefty fee divulges it as $2 : -1$. What does this mean? Take your million, he says, borrow another million in US funds (this makes $y = -1$), and put the 2 million into Canadian dollars (so $x = 2$). Lo and behold: Ahorita gets 2.22 for x and -1.11 for y after one year, the same 11% in both components and, of course, for the total. Again the ratio is maintained, and the same fabulous return will swell her account year after year.

Ms. Vengo's first impulse is to take all her money out of Poco's hands and give it to Mucho, but then she remembers her family ties and also the nasty law which forbids her making a net debt in a foreign currency. At least she is pleased with the neatness of the new arrangement. Instead of getting confused in the vagaries of *two currencies*, she now thinks in terms of *two portfolios*: Mucho's consistently grows at 11% and Poco's at 3%. It is as if she were working in a simpler monetary system: Mucho marks and Poco pounds.

What happened? Ahorita's brokers had stumbled onto the two ratios $2 : -1$ and $2 : 3$ defining the "eigenlines" of the matrix A. If we take vectors V and W from each of these lines and stick them together into a new matrix $M = [V, W]$ we get

$$AM = A \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1.11 & 0 \\ 0 & 1.03 \end{bmatrix} = MD\,,$$

because A affects V like multiplication by 1.11 (Loquiero's "eigenvalue"), and likewise W by 1.03 (Megusta's). The matrix M acts as a kind of dictionary from the new coordinates X' based on portfolios to the old ones $X = MX'$ based on currencies. If X is changed to AX, what happens to X'? Well, $AMX' = M(DX')$ says that X' has changed to DX'.

This would be clearer, if we had a matrix M^{-1} which undoes multiplication by M. Then $X = MX'$ would be the same as $M^{-1}X = X'$, in other words, M^{-1} would translate from old coordinates X to new coordinates X'. Multiplying $AMX' = M(DX')$ through by M^{-1}, we'd get $M^{-1}AMX' = DX'$, which (read right to left) says: *converting portfolio coordinates X' to currency ones MX', then applying the Mexican matrix A to obtain AMX', and finally reconverting to portfolio by applying M^{-1} is a waste of time, because it all just amounts to the easy diagonal multiplication DX'.*

In short, M^{-1} changes coordinates into a system which "diagonalises" the messy operation A. But does it exist? Sure: if we had enough matrix eperience, we would already know that

$$\begin{bmatrix} 3/8 & -1/4 \\ 1/8 & 1/4 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

but it's never too late to check it. The matrix with the fractions is M^{-1}, the "inverse" of M, and the one on the right hand side is, of course, the "identity". The whole equation is written as $M^{-1}M = I$. And yes: multiplying $AM = MD$ by M^{-1} formally yields $M^{-1}AM = M^{-1}MD = ID$, but the I quickly disappears, since multiplying by it changes nothing.

I conigli di Leonardo. In many biological species, population growth can be studied by keeping track of two types of females: *adults* and *juveniles*. Juveniles are those who were born in the last breeding season and survived to the present one. Presumably they produce less offspring than adults (often none at all) and survive less frequently to the next breeding season (in which they will be adults). Males are ignored in this kind of study, apart from being assumed available to fill out the reproductive capacities of the females.

Let x_n be the the number of adults, y_n the number of juveniles, and $t_n = x_n + y_n$ the total number of females (excluding babies) in the n-th season. What will be the situation in the following season? If a and b denote the survival rates for adults and juveniles, respectively, the number of adults will be $x_{n+1} = ax_n + by_n$. If every adult contributes (on the average) c juveniles to the next season, and every juvenile likewise contributes d future juveniles (usually $d = 0$), the younger set will number $y_{n+1} = cx_n + dy_n$. This can be summarized in the matrix equation

$$\begin{bmatrix} x_{n+1} \\ \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ \\ c & d \end{bmatrix} \begin{bmatrix} x_n \\ \\ y_n \end{bmatrix}.$$

Abbreviating the square matrix by A and the n-th "population vector" by X_n, this simplifies to $X_{n+1} = AX_n$, which could be jazzed up to $AX_n = A^2X_{n-1} = \cdots = A^nX_0$,

For any A with $a, b, c > 0$, and $d \geq 0$, the evolution of a non-negative population vector $X_n = A^nX_0$ happens to be governed by two facts:

1. As n increases, the fraction x_n/t_n approaches a constant x^* independent of the initial x_0, y_0. Of course, y_n/t_n will simultaneously approach $y^* = 1 - x^*$. We call this the *stable distribution* and any X with $x/t = x^*$ and $y/t = y^*$, a *stable vector*.
2. If X is a stable vector then so is AX. This means that, on a stable vector, multiplication by A has a very simple effect. Since the ratio between the components cannot change, both of them get multiplied by the same *growth factor* $\lambda > 0$. So, a stable X "experiences" A as if it were a simple number instead of a matrix: $AX = \lambda X$.

Taken together, these tell us that, if we wait long enough, X_n will approach stability and stay there. From then on our problem reduces to the familiar one-dimensional growth (or decay) of $t_n = x_n + y_n$ by the factor $\lambda > 0$. Our efforts must therefore be directed toward finding the stable distribution and the growth factor. Actually, we'll do this in reverse: first find λ and then solve the equation $(A - \lambda I)X^* = 0$ subject to the condition $x^* + y^* = 1$.

Example 1: For some species of birds, the reproduction story outlined above might yield a matrix with $a = .7,\ b = .3,\ c = 2,\ d = 0$. Repeated squaring yields

$$A^2 = \begin{bmatrix} 1.09 & 0.21 \\ \\ 1.40 & 0.60 \end{bmatrix} \quad A^4 = \begin{bmatrix} 1.48 & 0.35 \\ \\ 2.37 & 0.65 \end{bmatrix} \quad A^8 = \begin{bmatrix} 3.04 & 0.76 \\ \\ 5.05 & 1.27 \end{bmatrix}.$$

From the last power we see that the x-part of both columns seems to go toward $3/8 = .375$. This is borne out by checking A^{16}, the square of A^8. Thus $x = 3, y = 5$ should give a stable vector. Indeed multiplication by A yields $(.7)(3) + (.3)(5) = 3.6$ and $(2)(3) + (0)(5) = 6$, which is in the same ratio $3 : 5$ but multiplied by $\lambda = 1.2$

Example 2: One of the earliest " modern" European mathematicians (ca. 1200 A.D.) was Leonardo da Pisa, alias Fibonacci, the man who introduced Indo-Arabic numerals to the west. Today he is remembered mainly for the numerical sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots,$$

generated by the powers of the matrix A which has $a = b = c = 1$, and $d = 0$. It is remarkable that Leonardo actually posed his original problem in the context of the reproduction of rabbits (*conigli* in Italian), immortal ones with a survival rate = 1. Here we have

$$A^4 = \begin{bmatrix} 5 & 3 \\ \\ 3 & 2 \end{bmatrix} \quad A^8 = \begin{bmatrix} 34 & 21 \\ \\ 21 & 13 \end{bmatrix} \quad A^{16} = \begin{bmatrix} 1597 & 987 \\ \\ 987 & 610 \end{bmatrix}.$$

In the 16-th power the ratios for the first and second columns are both equal to 1.61803. This also happens to be the growth factor, whose theoretical value is $\lambda = (1 + \sqrt{5})/2$.

A Long Leap Forward. The most obvious method for finding λ is to look at increasing powers of A until we see the same ratio jelling in both columns. To get high powers of A quickly, it is advisable to square over and over, forming $A^2 = AA, A^4 = (A^2)^2, A^8 = (A^4)^2, A^{16} = (A^8)^2$, and so on, as we have done above.

We'll soon discover a way of directly computing the growth factor (a.k.a. "eigenvalue") as

$$\lambda = \frac{1}{2}\big[(a + d) + \sqrt{(a - d)^2 + 4bc}\big]. \tag{1}$$

by solving a quadratic equation. Of course, such equations usually have two solutions, the second one being obtained by replacing the + in front of the root by a minus sign. We would get -0.5 in the first example and $(1 - \sqrt{5})/2$ in the second, neither of which would make much sense in terms of the model. However, they do allow a formal diagonalisation, which we shall now explore for the case of Example 2.

Putting $(1-\sqrt{5})/2 = \mu$, following the prescriptions for finding eigenlines, and forming M, we get

$$AM = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda & \mu \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} = MD\,,$$

which is so slick and tidy because $\lambda + \mu = 1$ and $\lambda\mu = -1$. Together with $\lambda - \mu = \sqrt{5}$, these relations also allow us to verify

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\mu \\ -1 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & \mu \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which shows that the first of these matrix factors (*with* $\sqrt{5}$ in the denominator) is M^{-1}. The point of all this is to demonstrate another use of diagonalisation: computing powers. Note that $(M^{-1}AM)^2 = M^{-1}AMM^{-1}AM = M^{-1}A^2M$ because the $M^{-1}M$ in the middle wipes out. And so it goes for *all* powers: $(M^{-1}AM)^n = M^{-1}A^nM = D^n$. *To know the* $(n+1)$*st number in the Fibonacci sequence*, for instance, you can figure out that it will be top left entry of A^n. You then use your pocket calculator to find D^n and translate back by $A^n = MD^nM^{-1}$, like so:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} \lambda & \mu \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} \begin{bmatrix} 1 & -\mu \\ -1 & \lambda \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\,.$$

For the first entry of the result, you'll get $(\lambda^{n+1} - \mu^{n+1})/\sqrt{5}$. Since this has to be an integer and since $|\mu^n|$ is small for sizable n (for $n = 16$, it is less than 0.0005), the rule is this: *just compute* $\lambda^{n+1}/\sqrt{5}$ *and take the nearest integer.* For $n = 16$, the calculator says 1596.99987, which agrees nicely with the 1597 we got in Example 2 above.

3. More Algebra

The time has come to stop the hand-waving and get a little more precise. We shall do the minimum to give clean explanations of what happened in Section 2 and motivate what we are about to do in Section 4.

Inverses. The easy equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\,I \qquad (2)$$

turns out to be a gold mine of information. The second factor appearing in it is visibly concocted from the first factor A, and is known as its *adjoint*, denoted $A^\star$. The scalar $(ad-bc)$ on the right hand side is called the *determinant*, $\det A$. Equation (2) can thus be abbreviated as $AA^\star = (\det A)I$, and one easily checks that $A^\star A = (\det A)I$, as well.

If $\det A \neq 0$, the matrix A has an *inverse*

$$A^{-1} = \frac{1}{\det A} A^\star \qquad \text{with} \qquad AA^{-1} = A^{-1}A = I.$$

If A has an inverse, any equation $AX = 0$ clearly implies $X = 0$. On the other hand, if A does *not* have an inverse, then $AA^\star = 0$, and $AX = 0$ when X is either column of $A^\star$. Hence we can say that

A has an inverse if and only if $AX = 0$ is impossible with $X \neq 0$.

It should also be noted that inversion reverses the order of factors, i.e. $(AB)^{-1} = B^{-1}A^{-1}$, because $ABB^{-1}A^{-1} = AIA^{-1} = I$.

Rewriting $AA^\star = (ad-bc)I$ with $A^\star = (a+d)I - A$ yields the *Cayley-Hamilton relation*

$$A^2 - (a+d)A + (ad-bc)I = 0. \tag{3}$$

The scalar $(a+d)$ is called the *trace* of A and denoted by $\operatorname{tr} A$.

Computing the determinant of the product AB, we obtain the miraculous result
$(ax+by)(cu+dv) - (cx+dy)(au+bv) = (ad-bc)(xv-yu)$, i.e. :

$$\det(AB) = (\det A)(\det B). \tag{4}$$

Eigenstuff. A 2-column $V \neq 0$ is an *eigenvector* of A if multiplication by A only scales it by some numerical value λ, i.e., $AV = \lambda V$ or $(A - \lambda I)V = 0$ An eigenvector never comes alone. If $AV = \lambda V$ then also $AW = \lambda W$ for every scalar multiple $W = \alpha V$. Hence the whole "eigenline" determined by V experiences the action of A as a simple scaling by the *eigenvalue* λ.

There are several ways of computing eigenvalues. For 2×2 matrices they are easily obtained by applying the Cayley-Hamilton relation (3) to a putative eigenvector V and obtaining $(\lambda^2 - (\operatorname{tr} A)\lambda + \det A)V = 0$. Since V must be $\neq 0$ to qualify, must satisfy the *characteristic* equation $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$, which simplifies to

$$(\lambda - r)^2 = q \qquad \text{with} \qquad \begin{aligned} 2r &= a+d, \\ 4q &= (a-d)^2 + 4bc\,, \end{aligned} \tag{5}$$

and has at most two solutions, namely $r \pm \sqrt{q}$. This explains Equation (1). Once these λ are determined, it is easy to solve $(A-\lambda I)V = 0$ for the actual eigenvectors V. Note: $\operatorname{tr} A$ is the sum, and $\det A$ is the product of $\lambda_1 = r + \sqrt{q}$ and $\lambda_2 = r - \sqrt{q}$.

The extreme case $A - \lambda I = 0$ needs no further clarification and will be excluded from the following discussion. Otherwise, three cases can occur:

(a) There might be two different eigenlines, generated by (say) V and W. The matrix $M = [V, W]$ is then invertible, because $MX = 0$ is impossible (with $X \neq 0$), as it would be of the form $aV + bW = 0$. Then we get a diagonalisation $M^{-1}AM = D$, as in the illustrations of the preceding section. This happens for $q > 0$.

(b) The eigenvectors of A form a single line. Of course, the effect of A then must be the same on every eigenvector. Therefore $\lambda_1 = \lambda_2$, and $q = 0$.

(c) There are no (real) eigenvalues and eigenvectors, i.e. $q < 0$.

If we are allowed to use *complex* numbers, even $q < 0$ yields two eigenvalues and hence a diagonalisation $A = MDM^{-1}$ with *complex* matrices M and D.

Similarity. The matrices A and B are said to be *similar* if there is an invertible matrix M such that

$$B = M^{-1}AM \quad \text{or} \quad A = MBM^{-1}.$$

It is easy to see that, if two matrices are similar to a third, they are similar to each other.

What about eigengadgets? If A and B are as above, $AMV = MBV$ readily shows that V is an eigenvector for B if and only if MV is one for A, with the *same* eigenvalue.

In particular, *similar matrices have the same trace and determinant.* If you wish to avoid complex numbers completely, you can deduce this by applying M and M^{-1} to the Cayley-Hamilton relation, and see that it does not change. Hence similar matices will fall into the same one of the patterns (a), (b), or (c) outlined above.

Given any A, we shall now show a way of constructing an invertible matrix M such that the resulting $B = M^{-1}AM$ takes on one of the following three standard forms:

$$\text{(a)} \quad \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix} \qquad \text{(c)} \quad \begin{bmatrix} r & -s \\ s & r \end{bmatrix}. \tag{6}$$

In the diagonalizable case (a), we already know what to do: take the columns of M to be eigenvectors of A. In the other two cases we shall exploit the Cayley-Hamilton relation, written in the form

$$(A - rI)^2 = qI, \tag{7}$$

with r and q as in (5). If there is just one single eigenline ($q = 0$ and $r = \lambda$), choose $W \neq 0$ to lie outside it, so that $V = (A - rI)W \neq 0$. Then $(A - rI)V = 0$ because of (7), and V is an eigenvector. Putting $M = [V, W]$, we have $(A - rI)M = [0, V] = MJ_2$, where J_2 is the 2×2 matrix with 1 in the upper right and 0 elsewhere. J_r is exactly what you get by subtracting rI from the matrix in the middle of (6). The invertibility of M is ensured by the fact that its columns are not aligned, making $MX = 0$ impossible (for $X \neq 0$) and $\det M \neq 0$ (cf. the pargraph on inverses).

The case (c) really belongs under the heading of commplex diagonalisation, and is included here in its present form only for the sake of completeness. Grab any vector $Y \neq 0$ (there *are* no eigenlines to stay away from), and put $M = [V, W]$ with $W = sY$, where $s^2 = -q$, and $V = (A - rI)Y$. Then $(A - rI)M = [qY, sV] = s[-W, V]$. The rest is exercise.

4. The Jordan Decomposition

Over any field of scalars which contains all roots of the minimal (or characteristic) polynomial of an $n \times n$ matrix A, it is fairly straightforward to find an upper triangular matrix B similar to A. The final step, valid over *any* field, is to go from there to the "Jordan Form".

Jordan blocks. For every integer $r > 0$, let J_r denote the $r \times r$-matrix obtained by augmenting the $(r-1)$-st identity matrix I_{r-1} by a trivial first column and last row (and taking $J_1 = 0$). Any matrix of the form $\lambda I_s + J_s$, where λ is a scalar, will be called the *Jordan block of degree s and proper value λ.* As an example, consider

$$\lambda I_4 + J_4 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}. \tag{8}$$

Up to similarity, such matrices turn out to be the building blocks for *all* upper triangular matrices, in the following sense.

Theorem: *Any upper triangular $n \times n$ matrix A is similar to a "direct sum" of Jordan blocks $A_0, \ldots, A_l$, meaning that*

$$M^{-1}AM = \begin{bmatrix} A_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_l \end{bmatrix} \tag{9}$$

for suitable $n \times n$ matrix M.

Prelude. Before jumping into the belly of the proof, let us clear the deck of routine material.

1. Preparing an induction. Since M would commute with any cI_n, replacing A by $A - a_{11}I_n$ does not really change the problem, and we may therefore assume that the first column of A is zero. To the right of this column and below the first row of A, we see an upper triangular $(n-1)\times(n-1)$ matrix B, which by induction we may suppose to have already transfomed into a direct sum of Jordan blocks $B_1, \ldots, B_m$. Altogether we have

$$\begin{bmatrix} 1 & 0 \\ & \\ 0 & N^{-1} \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ & \\ 0 & N \end{bmatrix} = \begin{bmatrix} 0 & R_1 & R_2 & \cdots & R_m \\ 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_m \end{bmatrix} = A', \tag{10}$$

where N is the $(n-1)\times(n-1)$ matrix which had brought B into the desired form. At this point, the first column of our A' matrix is still 0, and the first row has the form $\left[0, R_1, \ldots, R_m\right]$, with the subrow R_k sitting above the block B_k. We shall need to remove these "obstructions" R_k.

2. Permuting the blocks. Whenever we permute the B_k by suitable similarities of A', the corresponding R_k are permuted accordingly. This follows from the fact that permutations are just reshuffling the index set with which rows and columns are labelled. As the index 1 is not affected by the shuffle, the first column stays as it is (namely zero), whereas the first row is rearranged. If that sounds too vague, you may wish to verify the following equation:

$$\begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_t & 0 \\ 0 & I_u & 0 & 0 \\ 0 & 0 & 0 & I_s \end{bmatrix} \begin{bmatrix} * & X & Y & * \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_u & 0 \\ 0 & I_t & 0 & 0 \\ 0 & 0 & 0 & I_s \end{bmatrix} = \begin{bmatrix} * & Y & X & * \\ 0 & B & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & * \end{bmatrix}. \tag{11}$$

Since every permutation can be achieved by systemeatically switching adjacent elements (think of books on a shelf), this apparently special case establishes our claim.

The key lemma. Left multiplication by J_r will push the entries of a column upward, losing the first and replacing the last by 0, while right multiplication by it will push the entries of a row to the right, losing the last and replacing the first by 0. These properties combine to yield the following magic, with $b + b' = c + c' = 0$ for typographical reasons:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b' & c' & 0 \\ a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} - \begin{bmatrix} b' & c' & 0 \\ a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

LEMMA: *Let $r > s$, and consider any $r \times s$ matrix Z whose non-zero entries are all in the first row. Then there exists an $r \times s$ matrix X such that $J_r X - X J_s = Z$.*

Proof. Equation (12) shows how X can be constructed: if Z has $[a_1\, a_2\, \dots\, a_s]$ as its first row, we use a_1 for the "subdiagonal" entries of X and $[-a_2\, \dots\, -a_s\, 0]$ for the first row.

Proof of theorem. The main tool for "killing" the obstructions R_k is the easily verified formula

$$\begin{bmatrix} I_t & -X \\ 0 & I_u \end{bmatrix} \begin{bmatrix} T & Y \\ 0 & U \end{bmatrix} \begin{bmatrix} I_t & X \\ 0 & I_u \end{bmatrix} = \begin{bmatrix} T & Z \\ 0 & U \end{bmatrix} \quad \text{with} \quad Z = Y + TX - XU\,, \tag{13}$$

which is valid for square matrices T and U of degrees t and u, respectively, and $t \times u$ matrices X, Y, Z. Taking the middle factor on the left to be the identity matrix yields $Z = 0$, and shows that this is indeed a similarity relation. It will be used in both steps of the proof.

Step 1. To begin with, we only reduce the obstructions in the most obvious way. In (13), we take $U = B_m$, whence Y is zero below the first row (which is $= R_m$). Then T must be the $(n-u) \times (n-u)$ submatrix of A' lying "north-west" of B_m. Setting the last $t-1$ rows of X to zero we get $TX = 0$ because of the zeroes in the first column of T. Thus we obtain $Z = Y - XU$, which really says $Z_* = R_m - X_* B_m$, where X_* and Z_* denote first rows, the remaining ones being zero.

If B_m is invertible, Z_* can clearly be made zero by a suitable X_*. If *not*, B_m must be $= J_u$, and $X_* B_m$ can be made into any u-tuple with 0 as first coordinate. Hence Z_* can always be reduced to having at most one non-zero entry, namely, the first one. If $Z_* = 0$ for any reason, our game is won by induction, because we can consider the puzzle solved for the smaller matrix T. Since we are allowed to switch B_m with any B_k, we can apply the same argument to all of them.

Step 2. If the game is not yet over, we are faced with a situation which still looks like (10), but where every R_k is of the form $[c_k, 0, \dots, 0]$ with $c_k \neq 0$. Moreover, each $B_k = J_{d(k)}$ is a Jordan block with proper value 0. Let us arrange them in decreasing size, and make sure that $R_1 = [1, 0, \dots, 0]$, by multiplying the first row of A' by $1/c_1$ (and its trivial first column by c_1, leaving it unaffected but producing a correct similarity). Now comes the surprise: the upper left corner of A' looks like

$$\begin{bmatrix} 0 & R_1 \\ 0 & J_{d(1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = J_r\,, \tag{14}$$

which is none other than $J_{d(1)+1}$. The matrix displayed in (10) has thereby morphed into

$$A'' = \begin{bmatrix} J_r & R_2 & \cdots & R_m \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_m \end{bmatrix}, \tag{15}$$

where $B_k = J_{d(k)}$ for $k > 1$, as before. With $r = d(1) + 1$ strictly greater than the other degrees $d(k)$, we can unleash the key lemma via the following expanded version of (13):

$$\begin{bmatrix} I_s & 0 & -X \\ 0 & I_t & 0 \\ 0 & 0 & I_u \end{bmatrix} \begin{bmatrix} S & R & Y \\ 0 & T & 0 \\ 0 & 0 & U \end{bmatrix} \begin{bmatrix} I_s & 0 & X \\ 0 & I_t & 0 \\ 0 & 0 & I_u \end{bmatrix} = \begin{bmatrix} S & R & Z \\ 0 & T & 0 \\ 0 & 0 & U \end{bmatrix}, \tag{16}$$

where $Z = Y + SX - XU$. For $S = J_r$ and $U = J_{d(m)}$, the lemma shows that $Y = R_m$ can be annihilated by a suitable choice of X.

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