Practice Test 2 Solutions

Math 311 Fall 2006
Test 2 (100 points)

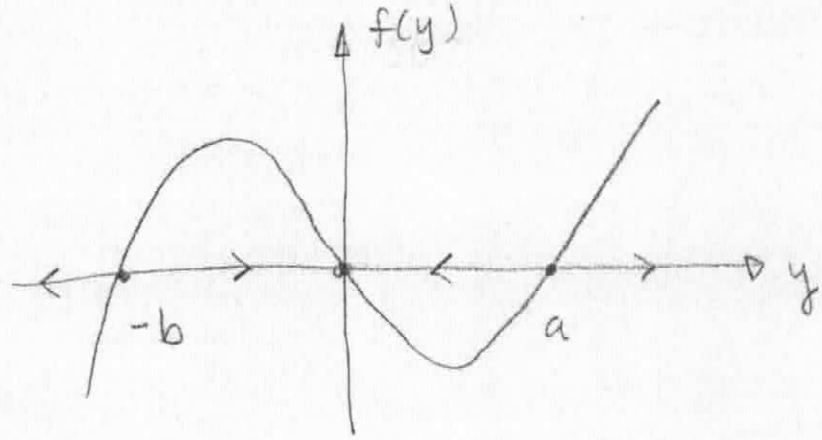
Fall 2007

Name: KEY

Instructions: You may use your graphing calculator. Work neatly. *Show at least one step of your work for full credit.*

1. (15 pts.) **Equilibrium Solutions and Stability**. Find the equilibrium solutions and evaluate their stability for the ODE:

$$y' = (y-a)(y+b)y; \quad a, b > 0$$
$$= f(y)$$



By the evidence of the graph f(y)

$x = 0$ is a stable equilibrium

$x = a$ & $x = -b$ are unstable equilibria

2. (15 pts.) **Numerical Approximation** . Using Euler's method compute an approximation to $y(1)$ when $y(t)$ solves the ODE

$$y' = -3y; \quad y(0) = 1.$$

Try step sizes equal to 0.25, 0.5 and 1.0. Comment on the differences you observe in the value $y(1)$. Compute the exact solution and evaluate the error in each case.

```
> f:=(x,y)->-3*y;
```

$$f := (x, y) \to -3\, y$$

```
> x0:=0: y0:=1: xf:=1:
> n:=1:  h:=evalf((xf-x0)/n);
```

$$h := 1.$$

Maple program for Euler's method

```
> x[0]:=x0: y[0]:=y0:
```

Take home message: Error decreases with step size

```
> for i from 1 to n do   k:=f(x[i-1],y[i-1]): y[i]:= y[i-1]+h*k:
  x[i]:= x[i-1]+h:  print(x[i],y[i]); od:
```

$(1., -2.)$ $= (x_1, y_1)$

```
> y[n];
```

e^{-3} is solution at 1

-2.

$e^{-3} = 0.04978$ nowhere near -2!

```
> evalf(exp(-3))-y[n];
```

2.049787068 error with $h = 1.0$

```
> n:=2:  h:=evalf((xf-x0)/n); for i from 1 to n do
  k:=f(x[i-1],y[i-1]): y[i]:= y[i-1]+h*k: x[i]:= x[i-1]+h:
  print(x[i],y[i]); od:
```

$x_0 = 0.0;\ y_0 = 1.0$

$h := 0.5000000000$

x_1 0.5000000000, -0.500000000 y_1

x_2 1.000000000, 0.2500000000 y_2

```
> evalf(exp(-3))-y[n];
```

-0.2002129316 error with $h = 0.5$

```
> n:=4:  h:=evalf((xf-x0)/n); for i from 1 to n do
  k:=f(x[i-1],y[i-1]): y[i]:= y[i-1]+h*k: x[i]:= x[i-1]+h:
  print(x[i],y[i]); od:
```

$h := 0.2500000000$

0.2500000000, 0.2500000000

0.5000000000, 0.0625000000

0.7500000000, 0.01562500000

1.000000000, 0.00390625000

```
> evalf(exp(-3))-y[n];
```

0.04588081837 error with $h = 0.25$

3. (20 pts.) **Velocity and Acceleration Models.** A 2000 kg vehicle is guided along a track with velocity $v = v(t)$ subject to a retarding force of the form $-kv$ and with initial speed $v(0) = 10$ meters/second, and $k = 200.0$ kg/second. For ease in computation assume the initial position is 0.

a) What is the velocity at $t = 2$ seconds?

$$m\frac{dv}{dt} = -kv \quad \text{Newton's 2nd Law}$$

$$\frac{dv}{dt} = -\frac{k}{m}v = \frac{-200}{2000}v$$

$$\frac{dv}{dt} = -0.1v; \quad v(0) = 10 \Rightarrow v(t) = 10e^{-0.1t}$$

$$v(2) = 10e^{-0.2} = \underline{8.18 \text{ m/s}}$$

b) What is the upper limit (in meters) of the car's progress along the track?

$$x(t) = \int v(t)\,dt = \int 10e^{-0.1t}\,dt$$

$$x(t) = -100e^{-0.1t} + C$$

$$x(0) = -100 + C = 0 \Rightarrow C = 100$$

so
$$x(t) = 100 - 100e^{-0.1t}$$
$$= 100(1 - e^{-0.1t})$$

and

$$\lim_{t\to\infty} x(t) = 100 \quad \text{since } e^{-0.1t} \to 0 \text{ as } t \to \infty$$

4. (20 pts.) **Some short answer questions about numerical methods.**

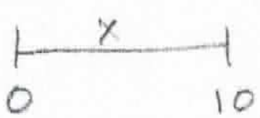
a) Why is the Runge Kutta numerical method we learned in section 2.6 sometimes called 4th order Runge Kutta?

Because the cummulative error is bounded by h^4, h being the step-size

b) How does the error scale with step-size for the Euler scheme?

$\text{error} \propto h^1$

c) If you wanted to use 100 iterations to estimate a solution, $y(x)$, to an initial value problem ($x = 0$), at $x = 10$, what would the step size have to be?



$$\frac{\text{Length}}{\#\ \text{iterations}} = \frac{10}{100} = 0.1$$

$$\boxed{h = 0.1}$$

d) Given a nicely behaved ODE, ideally you could decrease step-size to get any level of accuracy in the solution that you required. In reality you run up against the physical limitations of the computing machine you are using. There are two obvious ones that we discussed in class, what are they?

round-off error &

reasonable computation time.

5. (30 pts.) **Logistic Equation.** The number of supermarkets $C(t)$ at time t throughout the country that are using a computerized checkout system is described by the initial value problem

$$\frac{dC}{dt} = 0.0005C(2000 - C); \quad C(0) = 1.$$

where $t > 0$. How many supermarkets are using the computerized system when $t = 10$? How many companies are estimated to adopt the new procedure over a long period of time?

It's a logistic equation with $k = 0.0005$ and $M = 2000$. The solution is

$$c(t) = \frac{MC_0}{C_0 + (M - C_0)e^{-kMt}}$$

$$= \frac{2000}{1 + 1999e^{-t}}$$

25

$$c(10) = \frac{2000}{1 + 1999e^{-10}} \cong 1834 \text{ stores}$$

5

$$\lim_{t \to \infty} c(t) = M = 2000$$

30

6. Extra Credit (10 pts) In a series circuit containing only a resistor and an inductor, Kirchoff's second law states that the sum of the voltage drop across the inductor and the resistor is the same as the applied voltage $E(t)$. The voltage drop across the inductor is given by $L\frac{di}{dt}$, where $i(t)$ is the current and L is the inductance. The voltage drop across the resistor is equal to $R\,i(t)$, R being the resistance. For an R-L circuit then the current is governed by the following first order ODE:

$$L\frac{di}{dt} + Ri = E(t)$$

Solve this first order linear ODE when $R = 0.2$ ohms, $E(t) = 4$ volts, and $i(0) = 0$. Furthermore, assume the inductance is time dependent, and is governed by the following function:

$$L(t) = \begin{Bmatrix} 1 - \frac{t}{10} & 0 \le t < 10 \\ 0 & t \ge 10 \end{Bmatrix}$$

Hint: Solve two problems, one from $t = 0$ to 10, and the other from $t = 10$ onward.

I. $0 \le t < 10$: $\left(1 - \frac{t}{10}\right)\frac{di}{dt} + 0.2i = 4$

7

$$\frac{di}{dt} + \frac{2}{10-t}i = \frac{40}{10-t}$$ use integrating factors

$$i(t) = 20 + c(10-t)^2$$

$$i(0) = 20 + c \cdot 100 = 0 \Rightarrow c = -\frac{1}{5}$$

so $i(t) = 20 - (10-t)^2/5$ for $0 \le t < 10$

II. $t \ge 10$: $L = 0$ $\quad Ri(t) = E$

3

$$0.2\,i = 4$$

$$i = 20$$ (happily, since it makes $i(t)$ continuous)

$$i(t) = \begin{cases} 20 - (10-t)^2/5 & 0 \le t < 10 \\ 20 & 10 \le t \end{cases}$$