Math 311 Fall 2011
Final Exam (100 points)

Name: KEY

problem	points	score
1	10	
2	10	
3	16	
4	20	
5	20	
6	12	
7	12	
8 (E.C.)	10	
total	100+10	

Instructions: You may use your graphing calculator. Work neatly. *Show at least one step of your work for full credit.*

1. (10 pts.) Find the particular solution to this initial value problem.

$$\frac{dy}{dx} = 2y - 1; \quad y(1) = 1.$$

Separation of variables

2 pts $\displaystyle\int \frac{dy}{2y-1} = \int dx = x + C$

$u = 2y - 1$
$du = 2dy$

$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u = x + C$

$\frac{1}{2}\ln(2y-1) = x + C$ 2 pts

$\ln(2y-1)^{1/2} = x + C$

$(2y-1)^{1/2} = \sqrt{K}e^{x}$

$2y - 1 = Ke^{2x}$ 2 pts

$y_g = \frac{1}{2}(Ke^{2x} + 1)$

$y(1) = \frac{1}{2}(Ke^2 + 1) = 1$

$Ke^2 + 1 = 2$

$K = e^{-2}$

2 pts

$y = \frac{1}{2}(e^{-2}e^{2x} + 1) = \boxed{\frac{1}{2}(e^{2(x-1)} + 1)}$

2 pts

2. (10 pts.) Find the general solution of the following exact ODE.

$$\underbrace{(3x^2y^3+y^4)}_{M}\,dx+\underbrace{(3x^3y^2+y^4+4xy^3)}_{N}\,dy=0.$$ 2 pts

$$F_x = M = 3x^2y^3+y^4$$

$$F = x^3y^3+xy^4+f(y)$$ 2 pts

$$F_y = 3x^3y^2+4xy^3+f'(y)$$ 2 pts

$$F_y = N = 3x^3y^2+y^4+4xy^3$$

$$f'(y) = y^4$$

$$f(y) = \frac{y^5}{5}$$ 2 pts

$$F(x,y) = x^3y^3+xy^4+\frac{y^5}{5}$$ 1 pt

Implicit Solution:

$$\boxed{x^3y^3+xy^4+\frac{y^5}{5}=C}$$ 1 pt

3. (16 pts.) A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture, kept uniform by stirring, is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

2pts

$V = 1000\ L$ $\qquad r_{in} = 5\ L/s$

$x(0) = 100\ kg$ $\qquad r_{out} = 5\ L/s$

$\frac{dV}{dt} = r_{in} - r_{out}$ 2pts

$= 5 - 5 = 0$

so $V = 1000\ L$ for all t

4 pts

$$\frac{dx}{dt} = r_{in} C_{in} - r_{out} C_{out}$$

$C_{in} = 0$ (pure water)

$$C_{out} = \frac{x}{V} = \frac{x}{1000}$$

Solve $\frac{dx}{dt} = 0 - 5\frac{x}{1000}$ for $x(t)$

4 pts

$$\frac{dx}{dt} = -\frac{x}{200}$$

$$x(t) = x_0 e^{-t/200}$$

$$x(t) = 100 e^{-t/200}$$

Solve for t^*: $x(t^*) = 100 e^{-t^*/200} = 10$

$$e^{-t^*/200} = 1/10$$

4 pts

$$-t^*/200 = \ln 1 - \ln 10$$

$$\boxed{t^* = 200 \ln 10}$$

$$\boxed{t^* = 460.5\ \text{sec}}$$

4. (20 pts.) Consider the following nonhomogeneous second order linear ODE.

$$y'' + 4y' + 4y = 5xe^{-2x}$$

a) First write the associated homogeneous problem in operator notation and factor the operator.

5 pts

$$(D^2 + 4D + 4)y = 0$$

$$(D+2)(D+2)\,y = 0$$

$\underline{\text{or}}$ $\quad (D+2)^2\, y = 0$

5 pts b) Next find the solution to the associated homogeneous problem, y_c.

$(r+2)^2 = 0 \Rightarrow r = -2$ mult. 2

$$y_c = C_1 e^{-2x} + C_2 x e^{-2x}$$

5 pts c) Now find the annihilator of the right hand side $\underline{5xe^{-2x}}$.

$r = -2$ mult. 2

so $(D+2)^2$ is the annihilator

d) Use the annihilator to find the form of the particular solution, y_p. **DO NOT SOLVE FOR THE COEFFICIENTS.**

5 pts

$$(D+2)^2(D+2)^2\, y = (D+2)^2\, 5xe^{-2x}$$
$$= 0$$

solns are

e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}

Remove those appearing in y_c

$$y_p = Ax^2e^{-2x} + Bx^3e^{-2x}$$

5. (20 pts.) Consider the following set of two coupled first order linear constant coefficient ODEs.

$$x_1' = -x_1 + 4x_2$$
$$x_2' = 2x_1 - 3x_2$$

a) Write it as a matrix vector equation, $\mathbf{x}' = A\mathbf{x}$.

3 pts

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

7 pts

b) Compute the eigenvalues and eigenvectors of A.

1 pt

$$\begin{vmatrix} -1-\lambda & 4 \\ 2 & -3-\lambda \end{vmatrix} = (1+\lambda)(3+\lambda) - 8 = 0$$
$$= 3 + 4\lambda + \lambda^2 - 8 = \lambda^2 + 4\lambda - 5 = 0$$
$$= (\lambda+5)(\lambda-1) = 0$$

$\lambda = -5$

3 pts

$$\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2v_1 = -2v_2 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$v_1 = -v_2$$

So $\lambda = -5$ has e.vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 1$

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad -2v_1 = -4v_2$$
$$v_1 = 2v_2$$
$$\Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

3 pts

So $\lambda = 1$ has e.vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

c) From this write the vector valued general solution to the system.

2 pts

$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t}$$

8 pts d) Finally, find the solution with the initial condition $x_1(0) = 3,\ x_2(0) = 0$.

$$\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

So

$$c_1 + 2c_2 = 3$$

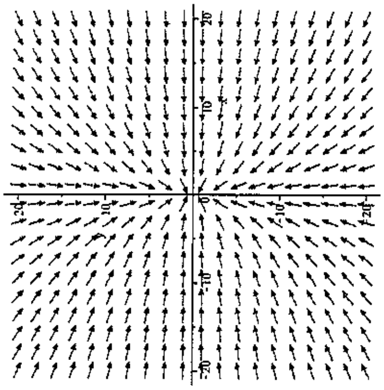
$$-c_1 + c_2 = 0 \Rightarrow c_1 = c_2$$

$$c_2 + 2c_2 = 3 \Rightarrow c_2 = 1$$

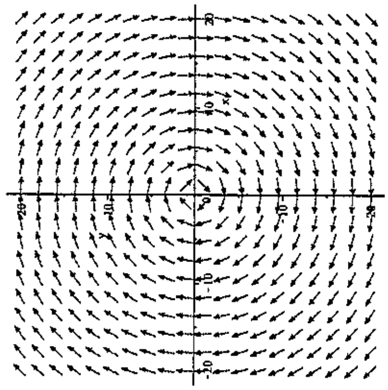
so $c_1 = 1$ also

$$\underline{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t}$$

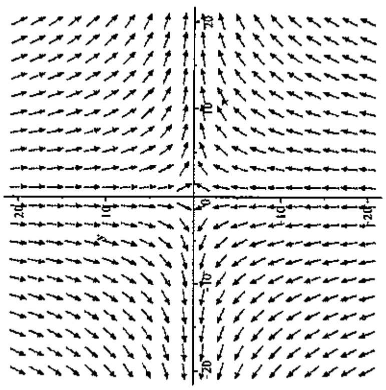
$$= \begin{pmatrix} e^{-5t} + 2e^{t} \\ -e^{-5t} + e^{t} \end{pmatrix}$$



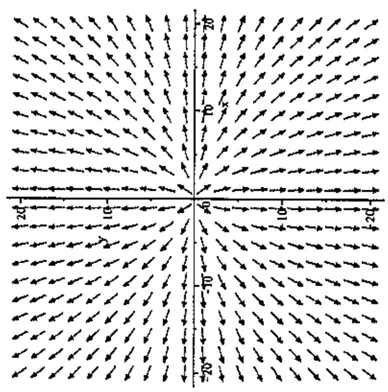
(a)



(b)



(c)



(d)

6. (12 pts.) For each of the following linear two dimensional systems, identify the correct phase plane (above), and determine the type and stability of the fixed point at the origin.

2 pts 1 pt.

3 pts a) $x' = 3x$
$y' = -2y$

$A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$; $\lambda = 3$ & $-2 \Rightarrow$ saddle (c)

3pts b) $x' = 5x$
$y' = 3y$

$A = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$; $\lambda = 5, 3 \Rightarrow$ unstable node (d)

3pts c) $x' = -2x$
$y' = -4y$

$A = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$; $\lambda = -2, -4 \Rightarrow$ stable node (a)

3pts d) $x' = y$
$y' = -x$

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\lambda = \pm i \Rightarrow$ center (b)

7. (12 pts) Circle *True* if the statement is true in all cases and *False* if it is not. Show work as needed for partial credit.

a) *True* *False* For the two-dimensional linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det\mathbf{A} \neq 0$, if the eigenvalues of $\mathbf{A}$ are real, unequal, and of the opposite sign, the origin $(0,0)$ is a saddle point.

saddle

b) *True* *False* The initial value problem $y' = 3xy^{1/3}$; $y(0) = a$ has a unique solution whenever $a \neq 0$.

x y

$f(x,y) = 3xy^{1/3}$ cont. for all x & y

$f_y = xy^{-2/3}$ not cont. at $y=0$

so long $y \neq 0$ there is a unique soln.

c) *True* *False* The mass-spring system governed by $x''+2x'+8x = \cos(3t)$ exhibits the phenomenon of beating.

No, it is a damped oscillator.

Beating only happens for an undamped forced oscillator

d) *True* *False* The equilibria for the autonomous system $x' = x - 2y + 5xy$; $y' = 2x + y$ are $(0,0)$ and $(1/2,-1)$.

$(0,0)$ obviously works

Solve $x - 2y + 5xy = 0$ → $x + 4x + 5x(-2x) = 0$

$2x + y = 0$ $x(5 - 10x) = 0$

$-2x = y$ $x = \frac{1}{2}$, so $y = -2(\frac{1}{2}) = -1$ ✓

e) *True* *False* The non-trivial equilibrium for the system in d) is an unstable spiral.

$\text{Jac} = \begin{pmatrix} 1+5y & -2+5x \\ 2 & 1 \end{pmatrix}$

$\lambda = \dfrac{-3+\sqrt{9+4.5}}{2}$

not complex

$\text{Jac}\Big|_{(1/2,-1)} = \begin{pmatrix} -4 & 1/2 \\ 2 & 1 \end{pmatrix}$; $\begin{vmatrix} -4-\lambda & 1/2 \\ 2 & 1-\lambda \end{vmatrix} = -(4+\lambda)(1-\lambda) - 1 = 0$

$-(4-3\lambda-\lambda^2) - 1 = \lambda^2+3\lambda-5 = 0$

f) *True* *False* In performing a numerical approximation to a solution of an ODE, decreasing the step-size decreases the accuracy of the approximation.

increases!

8. (10 pts) Extra Credit. Find the particular solution to the following initial value problem.

$$x_1' = x_1 - 2x_2$$

$$x_2' = 2x_1 + x_2$$

$$x_1(0) = 0; \quad x_2(0) = 4$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = 1 - 2\lambda + \lambda^2 + 4 = 0$$

$$= \lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} \quad -16$$

$$\lambda_{\pm} = 1 \pm 2i$$

$\lambda_+ = 1 + 2i$ find e.vector

$$\begin{pmatrix} 1-(1+2i) & -2 \\ 2 & 1-(1+2i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2v_1 - 2i v_2 = 0$$

$$v_1 = i v_2$$

let $v_2 = 1$, Then $v_1 = i$

$$\begin{pmatrix} i \\ 1 \end{pmatrix}$$

Find the real & Imag. parts of

$$\begin{pmatrix} i \\ 1 \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^t(\cos 2t + i \sin 2t)$$

$$= \begin{pmatrix} i e^t \cos 2t - e^t \sin 2t \\ e^t \cos 2t + i e^t \sin 2t \end{pmatrix} = \begin{pmatrix} -e^t \sin 2t \\ e^t \cos 2t \end{pmatrix} + i \begin{pmatrix} e^t \cos 2t \\ e^t \sin 2t \end{pmatrix}$$

$$Re = e^t \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}; \quad Im = e^t \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$$

$$X_g = e^t \left(c_1 \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} \right)$$

$$X(0) = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \Rightarrow \begin{matrix} c_2 = 0 \\ c_1 = 4 \end{matrix}$$

$$\boxed{X(t) = 4e^t \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}}$$