Math 311 Fall 2011
Test 2 (50 points)

Name: KEY

1. (6 pts) Circle *True* if the statement is true in all cases and *False* if it is not.

a) *True* *False* Two functions y_1, y_2 that are proportional to one another on an interval I are linearly independent on I.

If $y_1 = \beta y_2$, they are L.D.

b) *True* *False* The mass-spring system governed by $y'' + 2y' + y = 0$ is under-damped.

$r^2 + 2r + 1 = 0$

$r = \frac{-2 \pm \sqrt{4-4}}{2} = -1$, 2x $\Rightarrow$ critically damped

c) *True* *False* The annihilator for the function $x^2e^{-2x} + \cos 5x + x + 2$ is $(D-2)^3(D^2+25)D^2$.

$x+2$: $Q(D) = D^2$

$\cos 5x$: $Q(D) = D^2 + 25$

x^2e^{-2} : $Q(D) = (D+2)^3$

d) *True* *False* $x(t) = \cos 2t$ and $x(t) = \sin 2t$ are two linearly independent solutions to $x'' + 4x = 0$.

$r^2 + 4 = 0$

$r = \pm 2i \rightarrow x_g = C_1 \cos 2t + C_2 \sin 2t$

2 L.I. solns, from above

e) *True* *False* The Wronskian of the pair of functions $\cos 2t$ and $\sin 2t$ is zero for all t.

$$\begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2 \neq 0$$

f) *True* *False* The following 4th order constant coefficient D.E.s are equivalent, $y'''' + y''' + 2y' = 0$, is $D(D^3 + D^2 + 2)y = 0$.

$(D^4 + D^3 + 2D)y = 0$ or $D(D^3 + D^2 + 2)y = 0$

problem	points	score
1	6	
2	10	
3	12	
4	12	
5	10	
6 (E.C.)	3-5	
total	50 + 3-5 E.C.	

2. (10 pts.) *Constant Coefficient Linear ODE*

Consider

$$(2D^3 - 5D^2 + 2D)y = 0$$

$$2y''' - 5y'' + 2y' = 0$$

$$D(2D^2 - 5D + 2)y = 0$$

5 pts total

a) Find the general solution, $y(x)$.

from $Dy = 0 \rightarrow \{1\}$ 1pt

1pt

from $(2D^2 - 5D + 2)y = 0 \rightarrow \{e^{2x}, e^{x/2}\}$

$2r^2 - 5r + 2 = 0$

$r = \dfrac{5 \pm \sqrt{25 - 4 \cdot 2 \cdot 2}}{4}$

$r = \dfrac{5 \pm 3}{4} = 2, \dfrac{1}{2}$

2 pts

$$\boxed{y_g = c_1 + c_2 e^{2x} + c_3 e^{x/2}}$$

1pt

5 pts total

b) Find the solution when $y(0) = 3$; $y'(0) = 6$; $y''(0) = 9$.

$y(0) = c_1 + c_2 + c_3 = 3 \rightarrow c_1 = 3 - c_2 - c_3$

1 pt correct eq.'s

$y' = 2c_2 e^{2x} + \frac{1}{2} c_3 e^{x/2}$

2pts correct derivatives

$y'(0) = 2c_2 + \frac{1}{2} c_3 = 6 \rightarrow 4c_2 + c_3 = 12$

$c_3 = 12 - 4c_2$

$y'' = 4c_2 e^{2x} + \frac{1}{4} c_3 e^{x/2} = 9$

$y''(0) = 4c_2 + \dfrac{c_3}{4} = 9$

$16c_2 + c_3 = 36$

$c_3 = 36 - 16c_2$

So $12 - 4c_2 = 36 - 16c_2$

$12c_2 = 24$

$c_2 = 2$

$c_3 = 36 - 32 = 4$

1pt correct c's

and $c_1 = 3 - 2 - 4 = -3$

1pt

$$\boxed{y(x) = -3 + 2e^{2x} + 4e^{x/2}}$$

3. (12 pts.) *Damped Harmonic Oscillator*

Consider the mass-spring system, where the mass' position, $x(t)$, is governed by this ODE:

$$x'' + bx' + x = 0.$$

For each of the following values of b, **1)** find the general solution $x(t)$, and **2)** name the type of motion: *under-damped, over-damped, critically damped or undamped.*

a) $b = 3$

4pts

$r^2 + 3r + 1 = 0$

2pts

$r_\pm = \dfrac{-3 \pm \sqrt{9-4}}{2} = -\dfrac{3}{2} \pm \dfrac{\sqrt{5}}{2}$

$r_- < 0$ obviously, $r_+ < 0$ because $3 > \sqrt{5}$

(follows from oscillator problem, also)

$\boxed{x_g = C_1 e^{r_- t} + C_2 e^{r_+ t}, \quad \text{over damped}}$

1 pts 1pt

b) $b = 2$

$r^2 + 2r + 1 = 0$

$\dfrac{-2 \pm \sqrt{4-4}}{2} = -1$, mult. 2

$\boxed{x_g = (C_1 + C_2 t)e^{-t} \quad \text{critically damped}}$

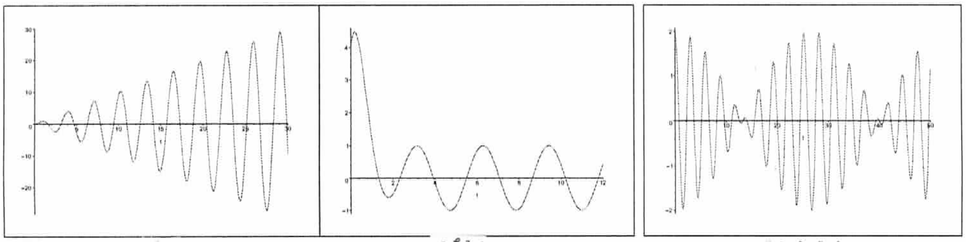
c) $b = 1$

$r^2 + r + 1 = 0$

$\dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

$\boxed{x_g = e^{-t/2}\left(C_1 \cos \tfrac{\sqrt{3}}{2}t + C_2 \sin \tfrac{\sqrt{3}}{2}t\right)}$

underdamped



(i) (ii) (iii)

4. (12 pts.) *Forced Oscillator*
Now consider the driven mass-spring system:

$$x'' + bx' + kx = \cos 2t.$$

For each of the following values of b and k find the solution to the unforced oscillator, x_c. Then match the solution to the forced oscillator to one of the graphs above. You shoud not solve for the particular solution to do this, in other words, there is no need to do undetermined coefficients.

a) $b = 4,\ k = 5$ $r^2 + 4r + 5 = 0$

2 pts $\frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$ $\boxed{(ii)}$ 1 pt

1 pt $\boxed{x_c = e^{-2t}(c_1 \cos t + c_2 \sin t)}$ transient dies away leaving $\cos(2(t-\delta))$

b) $b = 0,\ k = 5$

$r^2 + 5 = 0$

$r = \pm i\sqrt{5}$

$\sqrt{5} \neq 2$

but $\sqrt{5} \cong 2 \Rightarrow$ Beating

$\boxed{x_c = c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t}$ $\boxed{(iii)}$

c) $b = 0,\ k = 4$

$\omega_0 = 2 = \omega \Rightarrow$ resonance

$r^2 + 4 = 0$

$r = \pm i2$

$\boxed{(i)}$

$\boxed{x_c = c_1 \cos 2t + c_2 \sin 2t}$

5. (10 pts) *Undetermined Coefficients and Annihilators*
Find the general solution to this non-homogeneous second order constant coefficient ODE.

$$y'' + y = 10\, xe^{2x}$$

$P(D)\,y = 0$

$(D^2 + 1)\,y = 0$ associated homog. problem

$r^2 + 1 = 0$

$r = \pm i$ $\qquad y_c = C_1\cos x + C_2 \sin x$ $\quad$ 2 pts

$Q(D)$ annihilates xe^{2x} $\quad r = 2$, mult. 2 $\Big\}\,(D-2)^2$

$Q(D)P(D)\,y = (D-2)^2 (D^2+1)\,y = 0$

solution set $\{ e^{2x}, xe^{2x}, \cancel{\cos t}, \cancel{\sin t} \}$ $\leftarrow y_c$

Trial solution is $\quad y_p = Ae^{2x} + Bxe^{2x}$ $\quad$ 3 pts

plug into ODE:

$$y_p' = 2Ae^{2x} + Be^{2x} + 2Bxe^{2x}$$

$$y_p'' = 4Ae^{2x} + 2Be^{2x} + 2Be^{2x} + 4Bxe^{2x}$$

$$+\; y_p = +Ae^{2x} + Bxe^{2x}$$

$$(5A + 4B)e^{2x} + 5Bxe^{2x}$$

2 pts $\Big\{ = 0 \cdot e^{2x} + 10xe^{2x}$

$5B = 10 \Rightarrow B = 2$

$5A + 4B = 0$

1 pt $\quad 5A = -4(2) = -8$

$A = -8/5$

$$y_p = -8/5\, e^{2x} + 2xe^{2x} \quad \text{1 pt}$$

$$\boxed{y_g = y_c + y_p = C_1\cos x + C_2\sin x - 8/5\, e^{2x} + 2xe^{2x}}$$

1 pt

6. Do **only one** of the following *three* problems for extra credit.

Extra Credit (3 pts) *Theoretical*

Consider $P(D)y = 0$. What conditions on the roots of the characteristic equation would guarantee that every solution of the ODE satisfies:

$$\lim_{x \to \infty} y(x) = 0?$$

Assume $P(D)$ is order n

If Roots of characteristic eq. $(P(r)=0)$ are $(r_1, \dots, r_n)$, require $\mathrm{Re}(r_i) < 0$ for $i = 1, \dots, n$.

Extra Credit (5 pts) *Calculation*

A second order Euler equation is one of the form

$$ax^2y'' + bxy' + cy = 0 \quad (\star)$$

where a, b, c are constants. Show that if $x > 0$, then the substitution $v = \ln x$ transforms the ODE into the constant coefficient linear equation

$$a\frac{d^2y}{dv^2} + (b - a)\frac{dy}{dv} + cy = 0$$

with independent variable v. If the roots r_1 and r_2 of the characteristic equation for this ODE are real and distinct, show that a general solution of the Euler equation is $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

let $v = \ln x$ or $e^v = x$

then $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{dy}{dv}\frac{1}{x}$

& $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dv}\frac{1}{x}\right) = \frac{1}{x}\left(\frac{d^2y}{dv^2}\frac{1}{x}\right) - \frac{1}{x^2}\frac{dy}{dv}$

plug into $(\star)$:

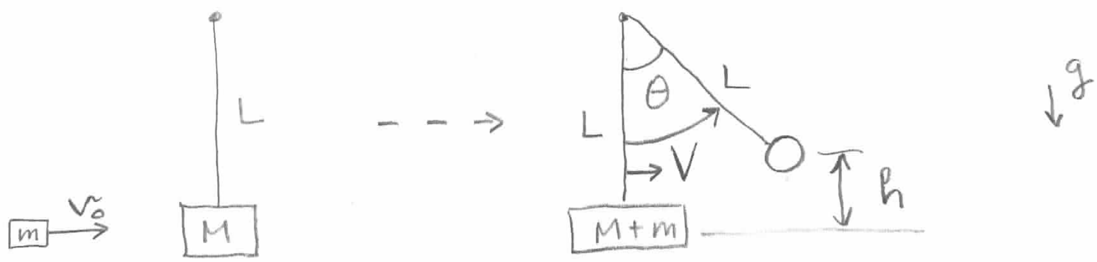
$$ax^2\left(\frac{1}{x^2}\frac{d^2y}{dv^2} - \frac{1}{x^2}\frac{dy}{dv}\right) + bx\frac{1}{x}\frac{dy}{dv} + cy = 0$$

$$a\frac{d^2y}{dv^2} - a\frac{dy}{dv} + b\frac{dy}{dv} + cy = 0$$

$$a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0 \quad \blacksquare$$

let roots $= r_1$ & r_2 & $y_g(v) = c_1e^{r_1v} + c_2e^{r_2v}$

So $y_g(x) = c_1e^{r_1\ln x} + c_2e^{r_2\ln x} = c_1x^{r_1} + c_2x^{r_2} \quad \blacksquare$



Extra Credit (5 pts) *Modeling*

In ballistics it is often important to be able to determine the muzzle velocity of the gun, that is, the speed of a bullet as it leaves the barrel. This can be determined indirectly with the aid of a ballistic pendulum (invented in 1742), which is simply a plane pendulum consisting of a rod of negligible mass to which a block of wood of mass M is attached. The system is set in motion by the impact of a bullet which is moving horizontally at the unknown velocity v_o. At the time of impact, which we shall take to be $t = 0$, the combined mass is $M + m$, where m is the mass of the bullet embedded in the wood. In the case of small oscillations the angular displacement $\theta(t)$ of a plane pendulum (see figure) is given by the equation for the harmonic oscillator $\theta'' + \frac{g}{L}\theta = 0$, where $\theta > 0$ corresponds to motion to the right of vertical. The velocity v_o can be found by measuring the height h of the mass $M + m$ at the maximum displacement angle θ_{max} shown in the figure. Intuitively, the horizontal velocity V of the combined mass (wood plus bullet) after impact is only a fractional amount of the velocity v_o of the bullet, that is $V = \frac{m}{m+M}v_o$. Recalling that a distance s traveled by a point along a circular path is related to the radius and central angle θ by $s = L\theta$, it follows that the angular velocity ω of the combined mass and its linear velocity v are related by $v = L\omega$. Thus the initial angular velocity ω_0 at the time of the bullet's impact is related to V by $V = L\omega_0$, in other words, $\omega_0 = \frac{m}{M+m}v_o/L$.

a) Solve the initial value problem

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \quad \theta(0) = 0, \quad \theta'(0) = \omega_0$$

b) Use the result to show that

$$v_o = \frac{M+m}{m}\sqrt{L\,g}\,\theta_{max}$$

c) Now express $\cos(\theta_{max})$ in terms of L and h, using the figure above and some trigonometry. Then use the first two terms of the Maclaurin expansion for $\cos(\theta_{max})$ to express θ_{max} in terms of L and h. Finally show that v_o is given (approximately) by

$$v_o = \frac{M+m}{m}\sqrt{2gh}$$

use back of exam

E.C. #3

a) $\theta_g = c_1 \cos\omega t + c_2 \sin\omega t$; $\omega = \sqrt{\frac{g}{L}}$

$\theta_g(0) = c_1 = 0$; $\theta_g'(0) = c_2\omega = \omega_0$

$c_2 = \omega_0/\omega$

$$\boxed{\theta_g(t) = \omega_0 \sqrt{\frac{L}{g}} \sin \sqrt{\frac{g}{L}}\, t}$$

b) Note that θ_{max} occurs when $\sin\sqrt{\frac{g}{L}}\, t = 1$,

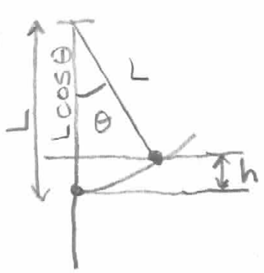
so from above, $\theta_{max} = \omega_0 \sqrt{\frac{L}{g}}$

Given $\omega_0 = \frac{m}{M+m}\frac{v_0}{L}$, so $\theta_{max} = \frac{m}{M+m}\frac{v_0}{L}\sqrt{\frac{L}{g}}$

Solve for v_0:

$$v_0 = \frac{M+m}{m} L \sqrt{\frac{g}{L}}\, \theta_{max}$$

or $$\boxed{v_0 = \frac{M+m}{m}\sqrt{gL}\, \theta_{max}}$$

c) 

$L - L\cos\theta = h$

$L(1-\cos\theta) = h$

$1 - \cos\theta = h/L$

$1 - h/L = \cos\theta$

if h is max amp. of swing

$$\boxed{1 - h/L = \cos\theta_{max}}$$

Maclaurin Series: $\cos\theta \cong 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$

So $\cos\theta_{max} \cong 1 - \frac{\theta^2_{max}}{2}$

Then $1 - h/L \cong 1 - \theta^2_{max}/2$

$$\sqrt{\frac{2h}{L}} \cong \theta_{max}$$

Plug into v_0 expression: $v_0 = \frac{M+m}{m}\sqrt{gL}\sqrt{\frac{2h}{L}}$

$$\boxed{v_0 = \frac{M+m}{m}\sqrt{2gh}}$$