

Final: Tuesday, December 15, 8AM.

1. Suppose that X has the following distribution function

$$F(x) = \begin{cases} 0, & \text{for } x \leq 0 \\ \frac{x}{8}, & \text{for } 0 < x < 2, \\ \frac{x^2}{16}, & \text{for } 2 \leq x < 4, \\ 1, & \text{for } 4 \leq x. \end{cases}$$

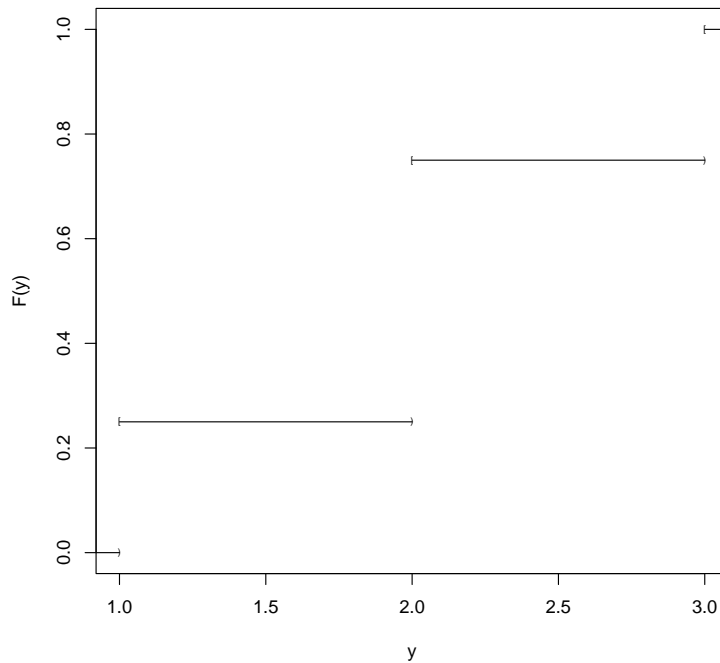
- a. Find p_{25} and p_{75} , the 25th and 75th percentiles of the distribution of X . Solving $F(p_{25}) = .25$ for p_{25} yields $p_{25} = 2$. Similarly, $p_{50} = 2\sqrt{2}$ and $p_{75} = 2\sqrt{3}$.
- b. Suppose that $Y = r(X)$ is defined by

$$r(x) = \begin{cases} 1, & \text{for } x \leq p_{25}, \\ 2, & \text{for } p_{25} < x \leq p_{75}, \\ 3, & \text{for } x < p_{75} \end{cases} \tag{1}$$

- c. Find the probability function of Y .

y	1	2	3
$P(Y = y)$.25	.5	.25

- d. Graph the distribution function of Y .



- e. Find a median of the distribution of Y . The smallest value of y satisfying $P(Y \leq y) \geq .5$ is the median. Hence, the median is 2.

2. Let $U \sim \text{unif}(0, 1)$.

- a. Compute EU and $E(U^2)$. $EU = .5$ and $E(U^2) = 1/3$
- b. Compute $\sigma^2 = \text{var}(U)$. $\sigma^2 = 1/3 - 1/4 = 1/12$.
- c. Compute $P(|U - \mu| \leq 2\sigma)$. First,

$$F(u) = \begin{cases} 0 & \text{for } u \leq 0, \\ u, & \text{for } 0 < u < 1 \\ 1, & \text{for } u \leq 1 \end{cases} \quad (2)$$

Then,

$$\begin{aligned} P(|U - \mu| \leq 2\sigma) &= F(\mu + 2\sigma) - F(\mu - 2\sigma) \\ &= F(1.077350) - F(-0.07735027) \\ &= 1 - 0 \end{aligned}$$

- d. According to Chebychev's inequality, $P(|U - \mu| \leq 2\sigma)$ is no less than what number? $1 - 1/4$

3. Suppose that $X \sim \exp(\lambda)$ with $\lambda = 1/10$.

- a. Compute EX . $EX = 1/\lambda = 10$
- b. What is the variance of X ? $\text{var}(X) = 1/\lambda^2 = 100$.
- c. Compute $P(|X - \mu| \leq 3\sigma)$. First, for $x > 0$, $F(x) = 1 - \exp(-\lambda x)$. Then,

$$\begin{aligned} P(|X - \mu| \leq 3\sigma) &= F(\mu + 3\sigma) - F(\mu - 3\sigma) \\ &= F(40) - F(-20) \\ &= 1 - \exp(-4) - 0 \\ &= 0.982. \end{aligned}$$

- d. According to Chebychev's inequality, $P(|Y - \mu| \leq 3\sigma)$ is at least what number? $1 - 1/9 = .889$.

4. Assume that the random variable Y has a distribution given by formula (2).

- a. Note the error in the practice exam: the values should be .1 and .3; hence the problem is: compute $P(.1 \leq Y \leq .3)$. The solution is

$$P(.1 \leq Y \leq .3) = F(.3) - F(.1) = .2 \quad (3)$$

- b. Let the events A and B be defined by $A = \{Y \geq .1\}$ and $B = \{Y \leq .3\}$. Write an expression for $A \cap B$ in terms of Y . $A \cap B = \{.1 \leq Y < .3\}$

- c. Compute $P(A \cap B)$.

$$P(.1 \leq Y \leq .3) = F(.3) - F(.1) = .2 \quad (4)$$

- d. Compute $P(Y \geq .1 | Y \leq .3)$.

$$P(Y \geq .1 | Y \leq .3) = \frac{P(A \cap B)}{P(B)} = \frac{P(.1 \leq Y \leq .3)}{P(Y \leq .3)} = .2/.3 \quad (5)$$