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ORIGINAL ARTICLE

Mathematical paradoxes as pathways into beliefs and polymathy: an experimental inquiry

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Abstract This paper addresses the role of mathematical paradoxes in fostering polymathy among pre-service elementary teachers. The results of a 3-year study with 120 students are reported with implications for mathematics pre-service education as well as interdisciplinary education. A hermeneutic-phenomenological approach is used to recreate the emotions, voices and struggles of students as they tried to unravel Russell's paradox presented in its linguistic form. Based on the gathered evidence some arguments are made for the benefits and dangers in the use of paradoxes in mathematics pre-service education to foster polymathy, change beliefs, discover structures and open new avenues for interdisciplinary pedagogy.

Keywords Beliefs · Interdisciplinarity · Paradoxes · Pre-service teacher education · Polymathy · Russell's paradox

1 Introduction

Elementary set theory serves as the backbone of mathematics content required by prospective elementary school teachers around the world. This is evident in the content standards of numerous curricular documents (e.g., Australian Education Council 1990; National Council of Teachers of Mathematics, 2000) which call for both a foundational and contextual understanding of the models for the four arithmetic operations ($+$, $-$, $\times$, $\div$) developed for the natural, whole, rational and real numbers. In addition the use of manipulatives such as Dienes base-10 blocks, Cuisenaire rods etc., greatly facilitate the enact-ion of the elementary arithmetic operations for the particular set under consideration. However the set theoretic and philosophical foundations of these operations are typically thought to be beyond the scope of pre-service education. The two fundamental questions explored in this paper are: (1) How can we facilitate the discovery of the mathematical foundations, paradoxes and structures? and (2) How can deeply rooted beliefs about the nature of mathematics be impacted?

2 Motivation and conceptual framework

In the United States, teacher professional development programs typically target in-service teachers and use an interventionist attempt to shift their beliefs and practices about the nature of mathematics. A large body of extant research addresses pre-service and practicing school teachers' beliefs and attitudes (e.g., Thompson, 1992) towards mathematics and describes the affective factors (Leder, Pehkonen & Törner, 2002) which influence mathematical understanding (Ball, 1990)and problem solving (Goldin, 2000, 2002). There is also research which addresses limitations of current research approaches to studying teacher beliefs (Leathman 2006; Wedege & Skott, 2006). Wedege & Skott (2006) argue that the main-stream trend of "research on belief-practice relationships runs the risk of becoming a self-fulfilling prophecy. It often contains a circular argument of claiming that certain observed mathematical practices are due to beliefs, while at the same

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time inferring mathematical beliefs from the very same practices." (p. 34). Similarly Leathman (2006) critiques research on teacher beliefs as assuming that "teachers can easily articulate their beliefs and that there is a one-to-one correspondence between what teachers state and what researchers think those statements mean. Research conducted under this paradigm often reports inconsistencies between teachers' beliefs and their actions." (p. 91).

Ernest (1989) categorized three philosophies of mathematics, namely the instrumentalist view, the Platonist view, and the problem solving view. The instrumentalist sees mathematics as a collection of facts and procedures which have utility. The Platonist sees mathematics as a static but unified body of knowledge. Mathematics is discovered, not created. The problem solving view looks on mathematics as continually expanding and yet lacking ontological certainty. The problem solving view sees mathematics as a cultural artifact. This implies that what is thought as true today, may not be seen as true tomorrow. (pp. 99–199). Ernest also describes absolutist and fallibilist views of mathematical certainty. The absolutist sees mathematics as completely certain and the fallibilist recognizes that mathematical truth may be challenged and revised (Ernest, 1991, p. 3).

Lerman (1990) recognizes that ones philosophy is related to ones preferred teaching style. The absolutist teacher will prefer a direct teaching style whereas a fallibilist is much more likely to engage in exploratory activities and open-ended problems. In what is now one of the seminal studies in the domain of teacher beliefs, Thompson (1984) studied three junior high teachers, all of whom had different beliefs about the nature of mathematics. The first teacher viewed mathematics as a coherent collection of interrelated concepts and procedures. She regarded mathematics as a subject free of ambiguity and emphasized conceptual development in the students. She would fit Ernest's model as a Platonist. The second teacher had a very different perspective of mathematics. Thompson says her teaching reflected more of a process-oriented approach than a content oriented approach. A view of mathematics as a subject that allows for the discovery of properties and relationships through personal inquiry seemed to underlie her instructional approach. This teacher would fit Ernest's model as person with the problem solving view. The third teacher in Thompson's study saw mathematics as a collection of facts and procedures which help students find the answer. She saw no ambiguity in mathematics. She would fit Ernest's model as a person with an instrumentalist view. Thompson sees at least three distinct ways of viewing mathematics, all of which greatly influence the choice of curriculum and its delivery. Thompson (1992) says that research on teachers cognitions and studies of teachers' conceptions have contributed to a conceptual shift in the field of research on teaching, moving away from a behavioral conception of teaching towards "a conception that takes account of teachers as rational beings" (p. 142). Our understanding of teaching from teachers' perspectives complements our growth of understanding of learning from learners' perspectives, which in turn, enriches the idea of schooling as the negotiation of norms, practices and meanings (Cobb,1988). Much earlier, Fenstermacher (1978) predicted that the single most important construct in educational research are beliefs (Pajares 1992). Törner & Sriraman (2007) argue that Thompson's (1992) theory to explain teacher's actions in a mathematics classroom based on their beliefs about mathematics is one instance of the development of a local philosophy based on problematizing research in the domain of beliefs. Thompson (1992) wrote.

> I think we will get further evidence on the role of teachers' views of mathematics when we go into more detail and investigate their understanding of different domains of mathematics, of specific components such as the meaning of mathematical concepts, proof, definition, theorem, conjecture, variable, symbols, rule, formula, axiom, problem, problem solving, application, model, computation, graphical representation, visualization, metaphor,etc., both with respect to the various sub-domains of mathematics as well as in a more general sense. (p. 142).

Today we usually speak of teachers' beliefs, which are generally formulated as ''views about mathematics'' (e.g. Grigutsch 1996; Pajares 1992). It is assumed that different beliefs about mathematics have different associated philosophies and/or epistemologies (Törner, 2002). Amidst all this important research which increases our understanding of teacher beliefs, in lieu of Thompson's call, one is left wondering about the dearth of studies (recent or otherwise) in describing or analyzing prospective elementary school teachers understanding of the foundational (set-theoretic) concepts of the mathematics they are exposed to. Could it be that the aftermath of New Math has had an "affective" impact (pun-intended) on the focus of mathematics education researchers engaged in teacher education and led to less emphasis of this particular mathematics content. For the author it is important that we attend to the significant role that tasks may have in teacher education, particularly tasks that foster interdisciplinary thinking as well as tasks that shake the dominant views of prospective teachers on the nature of mathematics. In the remainder of this paper the author will report on the use of a set-theoretic task to help prospective elementary school teachers understand and discover paradoxes and structures and foster polymathy. Based on the findings, limitations and dangers in such tasks are also outlined.

3 Polymathy, paradoxes and philosophy

The term polymath is in fact quite old and synonymous with the German term "Renaissance-mensch." Although this term occurs abundantly in the literature in the humanities, relatively few (if any) attempts have been made to isolate the qualitative aspects of thinking that adequately describe this term. Most cognitive theorists believe that skills are domain specific and typically non-transferable across domains. This implicitly assumes that "skills" are that which one learns as a student within a particular discipline. However such an assumption begs the question as to why polymathy occurs in the first place? (Sriraman & Dahl, 2008). The author argues that *polymathy* as a thinking trait occurs frequently in non-eminent samples such as high school students, and pre-service students, when presented with the opportunities to engage in trans-disciplinary behavior. In particular the use of unsolved classical problems and mathematics literature has been found to be particularly effective in fostering inter-disciplinary thinking (e.g Sriraman, 2003a, b, 2004, 2005).

Root-Bernstein (1989, 1996, 2000, 2001, 2003) has been instrumental in rekindling an interest in mainstream psychology in a systematic investigation of polymathy. That is the study of individuals, both historical and contemporary, and their interdisciplinary thinking traits which enabled them to contribute to a variety of disciplines. Common thinking traits of the thousands of polymaths (historical and contemporary) as analyzed by Root-Bernstein, Robert Sternberg, Dean Simonton and many others are: (1) Visual geometric thinking and/or thinking in terms of geometric principles, (2) Frequent shifts in perspective, (3) thinking in analogies, (4) Nepistemological awareness (that is, an awareness of domain limitations), (5) Interest in investigating paradoxes (which often reveal interplay between language, mathematics and philosophy), (6) Belief in Occam's Razor [Simple ideas are preferable to complicated ones], (7) acknowledgment of Serendipity and the role of chance, and (8) the drive to influence the Agenda of the times.

4 Methodology

In order to investigate whether prospective elementary school mathematics teachers display some of the thinking traits of polymaths the author conducted a 3 year study with approximately 120 prospective elementary school mathematics teachers in the 2002–2005 time period. These pre-service teachers were enrolled in the mathematics content sequence for elementary teachers at a large university in the western United States. This two semester content sequence is the only required mathematics content for these prospective teachers due to the particular state legislations in this region. The author was also the instructor of these students. Journal reflective writing was an integral part of this course.

During the course of the semester, among the various tasks assigned to the student was the following:

4.1 The task

> The town barber shaves all those males, and only those males, who do not shave themselves. Assuming the barber is a male who shaves, who shaves the barber? Explain in your own words what this question is asking you? When you construct your response to the question, please justify using clear language why you think your answer is valid? If you are unable to answer the question who shaves the barber, again justify using clear language why you think the question cannot be answered.

This task is the well-known linguistic version of Russell's paradox, appropriately called the Barber Paradox. In fact the task is actually classified as a less challenging paradox in the hierarchy of paradoxes that have baffled logicians, mathematicians and mathematical philosophers (Sainsbury, 1995). The question as formulated here was read out several times in the class in order to clarify what it was asking. Students were given about 10 days to construct a written response to this task. The purpose of this task was to investigate whether students with no prior exposure to the paradox would be able to decipher the contradictions in the linguistic version of Russell's paradox, and whether they would be able to then construct their own set theoretic (mathematical) version of the paradox. All the students were also asked to complete the following affective tasks parallel to the mathematical task. The students were also requested not to consult the worldwide web in search of a solution.

Write one paragraph (200–300 words) about your impressions of a given question after you have read it, while tackling it (if possible), and after you have finished it.

In particular record things such as:

The immediate feeling/mood about the question (confidence, in confidence, ambivalence, happiness, tenseness etc.)

After you've finished the question record the feeling/mood about the question (if you are confident about your solution; why you are confident? Are you satisfied/unsatisfied? Are you elated/not elated? Are you frustrated? If so why?

Did you refer to the book, notes? Did you spend a lot of time thinking about what you were doing? Or was the solution procedural (and you simply went through the motions so to speak).

Was the question difficult, if so why? If not, why not?

Did you experience any sense of beauty in the question and/or your solution?

4.2 Data collection and analysis

Out of the 120 students, 52 students were eventually able to unravel the paradox, i.e., understand and explain the contradiction in their writings, 40 students believed there was no contradiction (i.e., they answered that the barber shaved himself), and 28 students gave up on the problem but completed the affective portion. Over the 3-year period, in addition to the written journal responses of the students to the aforementioned tasks, the author interviewed 20 students from the 120 students who were representative of the larger sample.

Of these ten students had successfully unraveled the paradox (out of the 52), and the remaining ten were unable to unravel it. Six of these students came from the subset of 40 who saw no contradiction, and the remaining represented those who gave up on the problem. These students were purposefully selected on the basis of whether they were able to unravel the paradox and formulate its set theoretic or mathematical equivalent and those that were unsuccessful in their attempts to do so. It should be noted that all students were given full credit for the assignment irrespective of whether they were successful or not. The written artifacts (students journal writing/solution and affective responses), and interview data were analyzed using a phenomenological -hermeneutic approach (Merleau-Ponty, 1962; Romme & Escher, 1993) with the purpose of re-creating the voice of the students. Phenomenology has its roots in the philosophical work of Husserl and Heidegger, which was extended into a theory of embodiment by Merleau-Ponty in order to counter reductionism, dualism and to capture the totality of human experience.

During the interview students re-voiced their experience of unraveling the paradox. The author simply sought clarifications on the written solution, their affective responses and asked students for general comments on the nature of the problem and their struggles with it. This led to questions on their beliefs about mathematics. There was no pre-set direction or protocol in which the interview was made to progress. Each interview lasted approximately 60 min. The individual transcripts and the author's interpretation of student voices, particularly their self-reported affective and polymathic experiences was discussed with each student to ensure validity and reliability. Student journal writings were coded several times to categorize and determine the affective mood self reported by the students. Similarly the interview transcripts were also coded for affective categories and to determine consistency in the self reported voices. In addition, the constant comparative method from grounded theory was applied for the purposes of triangulating the categories which emerged from the phenomenological approach (e.g., Annells, 2006). Finally, in all classes a de-briefing session occurred in which various students presented their views on the problem and their solutions with a discussion of the contradiction.

5 Results

Examples of affective responses, student solutions, and interview transcripts are now presented in a phenomenological style, which recreate student voices as they struggled through this paradox. Contrasting voices are presented and are representative of the 20 students who participated in the interviews.

5.1 Voices

Note the following abbreviations:

Journal writing affective voice (JWAV) (response); Journal writing solution voice (JWSV) (response); Interview voice student (IVS); Interview voice author (IVA).

> JWAV1: Started the question in class (10/19) after you (author) read it. Stopped occasionally for work and classes, finished it 10/25 at 12.01 pm. The question is very confusing and I feel very anxious about this question. Why would this question be asked in a math class? Do not get me wrong, I like to think about questions like these, but they are difficult and time consuming. I'm frustrated…[w]hat a strange question.
>
> JWAV2: I painfully came to the conclusion that the question was answerable through reading the question several times and thinking about it for days. It is a beautiful paradox, if one thing were changed in the question I feel there would be a definite answer. The English in it is perfect.
>
> JWSR1: The question is asking who shaves the barber? However the barber is a male and he shaves only those males, who do not shave themselves. Thus he cannot shave himself because he only shaves the males that do not shave themselves…[B]ut the barber only shaves those who do not shave themselves…[T]his is a paradox, he cannot be the barber and shave himself, and he cannot shave himself and be the barber.
>
> JWSR2: The barber shaves himself. I justify this firstly by the opening sentence "who do not shave themselves" which implies there are those who do shave themselves. For example a mechanic will work on all of other peoples cars in town but if his car has a problem, he would work on it!

JWAV3: My immediate opinion was that the answer would be "yes". I didn't really feel anything when doing this problem other than that the answer was obvious.
JWAV4: This question was the death of me…[I] was more than upset with you. Thinking about this question made me so frustrated that I stopped and decided not to waste my time on it. I found this question to have not any beauty in it, all it caused me was a lot of stress and discomfort in my life. It is impossible to put any mathematical equation into it.

The affective response of frustration and curiosity as well as a sense of beauty in the problem was found predominantly in the group that unraveled the paradox (group 1). Students in this group voiced high levels of frustration, anger, curiosity and beauty as well as reported a sense of accomplishment in unraveling the paradox.

In contrast, in the groups that saw no contraction (group 2) and the group that gave up on the problem (group 3), although the levels of frustration were comparably high, the proportion of students that became curious or saw some beauty in the problem was considerably lower. Approximately half of the students in group1 and group 3 became angry while attempting this problem, however nearly 80% of group 1 students reported curiosity whereas only 10% of group 3 students experienced a sense of curiosity. The proportion of self reported experience of "sadness" was considerably higher in group 3 in comparison to the other groups. Group 1 reported the highest levels of accomplishment. Table 1 gives a summary of the coded affective voices/responses extracted from student journal writings and interview transcripts.

5.2 Student journal writings

In this section, sample journal writings with student solutions are presented (Fig. 1).

5.3 Commentary 1

This solution represents the numerous solutions obtained from group 1 students where the set-theoretic contradiction became somewhat clear to the students. It should be noted that students who discovered and formulated the set theoretic version of the solution spent significantly longer periods of time with the problem in comparison to the other groups. It should be noted that only 16 of the 52 students in group 1 were successful in doing this. It is well known in the foundational mathematics literature that Bertrand Russell discovered this paradox and communicated it to Gottlob Frege in a letter right when Frege was about to complete his treatise on the foundations of Arithmetic (*Grundlagen der Arithmetik*), which dealt a devastating blow to his work. The standard way of stating the paradox is if we let R be the set of all sets that are not members of themselves. Then R is neither a member of itself nor not a member of itself.

5.4 Commentary 2

There were four students in group 2 who first claimed that the barber shaved himself but then attempted to construct a linguistic explanation like the one shown in Fig. 2, which were incorrect. Two of these students were interviewed during which they voiced the reasons for doing so, which is reported in a following section of this paper (Fig. 3).

5.5 Voices 2

In this section, three interview vignettes are presented representing the three groups.

5.5.1 Interview vignette 1

IVS1: I was tickled by the challenge of this question. I did not expect to solve a paradox in a math class and had to

Table 1 A summary of the coded affective voices/responses extracted from student journal writings and interview transcripts

Student groups	Affective mood							
	Frustration	Anger	Curiosity	Sense of beauty	Sense of accomplishment	Set-theoretic formulation	Sadness	No affect reported
Group 1 (52) (unraveled paradox)	52	26	41	28	35	16	0	0
Interview subgroup 1 (10)	10	7	10	4	8	6	0	0
Group 2[a] (40) (Saw no contradiction- claimed the barber shaved himself)	32	11	14	0	3	0	1	8
Interview subgroup 2 (6)	3	1	1	0	2	0	1	1
Group 3 (28) (gave up on the problem)	27	13	3	0	0	0	14	1
Interview subgroup 3 (4)	4	2	1	0	0	0	3	0

[a] Four of the students in group 2 tried to construct an explanation to unravel the paradox after stating that the barber shaved himself (see Fig. 2)

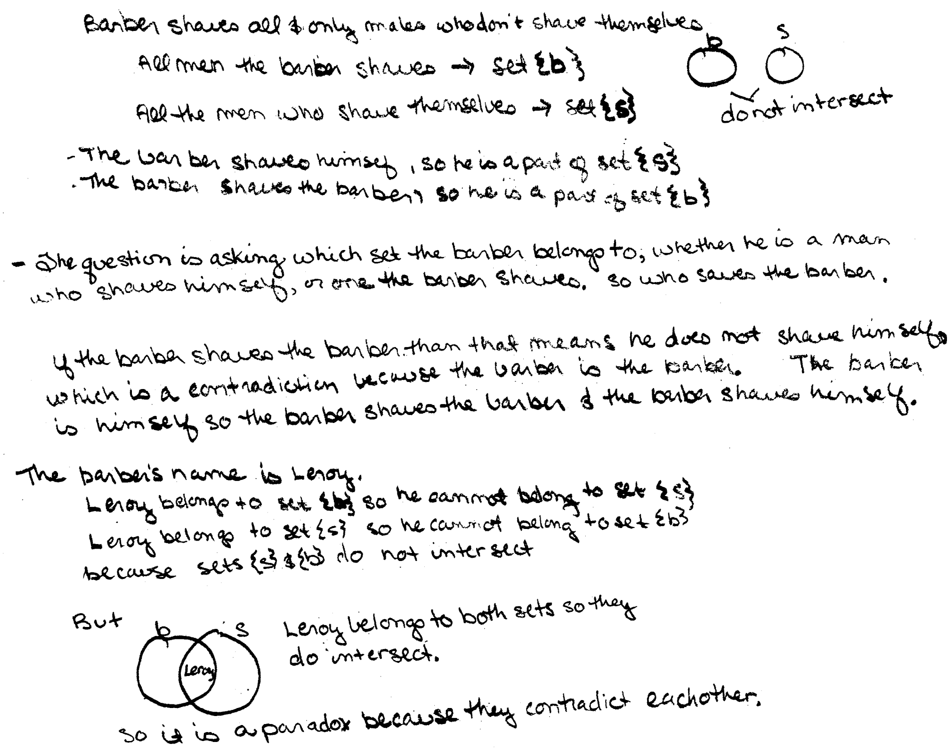

Fig. 1 Unraveling the set theoretic version of the paradox (group 1)

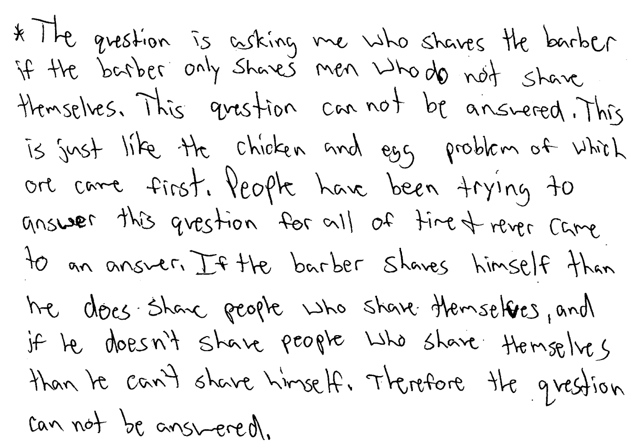

Fig. 2 Attempting to unravel the paradox linguistically

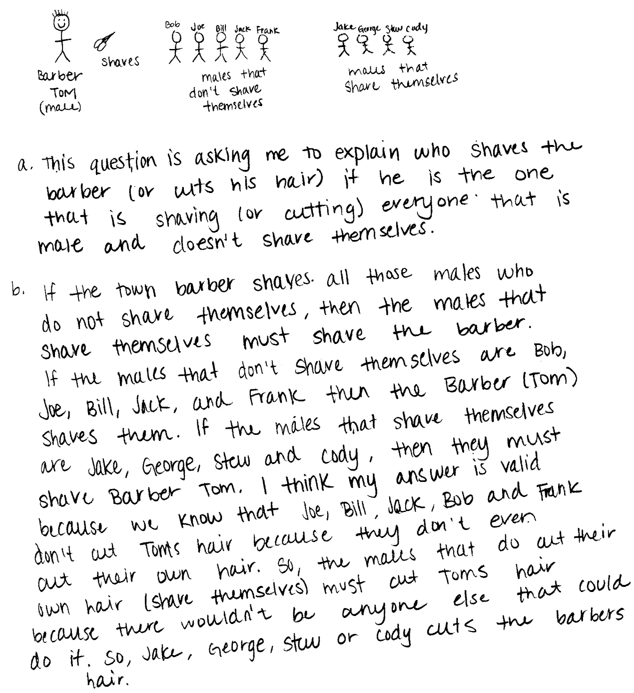

Fig. 3 Unsuccessful solution to the paradox

think a great deal when I was working on other things. I didn't see it as a problem involving sets at least not right away.

IVA1: Did you think it was a mathematical problem?
IVS2: Not really. It made me think of literature or something you would encounter from Zen Buddhism or even Rumi. Math problems don't make me angry and this one did! I was angry but challenged enough that I did not let my anger get in the way of finding a solution. I also thought about it in terms of art, like the strange tiling pattern posters we have in class, you know what I mean?

IVA2: Yes, I think you're referring to the Escher posters. How long did you spend on the problem?
IVS3: Yeah… that is the one. Days and days… [b]ut when I saw through it and realized it was a paradox, like you know assuming one thing led to another thing which

was contradictory, then I thought it was beautiful. I never imagined saying something like that for a math problem.
IVA4: Why not? So is this a math problem?
IVS4: Umm….in math classes, you like use formulas and equations and stuff and get answers. This one made you think really hard. I wish math were like that. Everything is just black and white in math, atleast this is what we are led to believe.
IVA5: Does this make you change your view of what mathematics is?
IVS5: This problem does, but I don't think we can say that for sure.
IVA6: What if I told you there were many such paradoxes in math? Would that make you see math differently?
IVS6: Sure. Look, we were told all through school that you got to memorize equations and formulas and things like that to do math or like there is a set way to do a problem. But with problems like these you kinda get a little from freedom to think for yourself. So definitely, yeah…if we were exposed to more paradoxes like these, we'd change our minds about what math is all about.

5.5.2 *Interview vignette 2*

IVA1: You wrote this problem caused the "death" of you? I am sorry you were so frustrated by the problem
IVS1: Yeah, I was really quite upset with it. I really thought about it for a while and became so confused that I didn't want to deal with it anymore.
IVA2: Would you have approached the problem differently if you were this in a non-math class?
IVS2: I actually did think about my philosophy course when I was reading this problem. But I am not used to this kind of stuff in a math for elementary teachers class. I did look into the book and it said that there were some paradoxes in math in the first chapter. It was easier to give up than go crazy over the problem. Plus I was really under pressure from work and just did not have the time to think it through. Sorry.
IVA3: No, no, you do not have to apologize. You simply did what you did. Did you really stop thinking about the problem after you gave up?
IVS3: [silence] Yes and no. I did not really feel like solving it. Part of the reason might have to do with that I did not really know what a solution is supposed to look like, you know. I did imagine other situations that could be like the one in the problem but it did not help any.
IVA4: Do you think paradoxes have a place in mathematics?
IVS4: I do not know. I am going to do my student teaching soon and I was thinking this might be a fun thing for kids to talk about. I do not know how it would go over with parents if the teacher gave problems like these [laughing].
IVA5: You mean they might get upset?
IVS5: Yeah…I mean they would try to help their kid and get a headache and then blame the teacher for giving this problem. Parents would want the answers to be cut and dry.

5.5.3 *Interview vignette 3*

IVA1: I noticed you first created an example with ten people and said it was clear that the barber had to shave himself. Then you expressed doubts after that and said it was like the chicken and the egg problem.
IVS1: My first feeling about the question was complete frustration. I did not know who or what to relate this question to. How was I supposed to answer the question? So, I did this problem solving thing like making an example. But then I was not really using the information given in the problem. Just making stuff up….[s]o I was not too happy about it and thought maybe I should do something else. That is when I wrote the stuff about the chicken and the egg problem. Now it seems clear to me that there was a contradiction in the way the problem was given. It was a good brainteaser but very frustrating though.
IVA2: Did you get upset?
IVS2: Honestly, yes. I got pretty worked up. I wasn't satisfied with the solution with people in it. It was all made up. The really upsetting thing was not knowing whether you were right or wrong.
IVA3: Is that how mathematics is supposed to be?
IVS3: [silence]
IVA4: By the way, did the problem make you think of other things?
IVS4: I read and re-read the problem many times and tried to think of it in different ways. I mean I thought of making stuff up like, the barber is married, his wife shaves him. It was tough to stay within the boundaries of the question. It made me think of puzzles or visual tricks that you see in paintings sometimes. Like you see it one way and then you blur your eyes or focus on a different point in the picture and you see something else. You know like those pictures that appear like a duck or a rabbit, or dolphins and people in the same picture. I guess the point of the problem was you could not have it both ways. I finally figured out that there were like two sets, one which had all men that shaved themselves, and then another set with all men shaved by the barber, and the barber could not belong to either of these sets.
IVA5: Are you happy you figured out the paradox later?
IVS5: I do not know about happy. More like relieved!

5.6 Author voice over

These three interview vignettes revealed that students were beginning to make connections between mathematics and other domains of inquiry such as philosophy, language and art. The hermeneutic analysis of student journal writings, affective responses and interview transcripts indicated that nearly half of the students (predominantly in group 1) displayed one or more *polymathic* traits when engaged with the paradox. In particular students reported (1) Frequent shifts in perspective (2) thinking with analogies, (3) tendency towards nepistemology i.e., questioning the validity of the question and its place in the domain of mathematics as well as the fallibility of mathematics. The pre-service teachers also reported an increased interest in the place of paradoxes in mathematics, which they had always believed as an infallible or absolutist science. Some of the students began to connect mathematics with language and voiced the need to engage in discourse as opposed to engaging in such an activity solitarily. One of the dominant and consistently heard student voices over the course of three years was of the deeply held dominant belief of mathematics as infallible or absolutist. Table 2 gives the polymathic traits displayed by the three groups of students.

6 Implications and concluding points

Polymathy and interdisciplinarity are topics on which one finds scant literature in the field of mathematics education, particularly in domain of pre-service elementary teacher education. Although we live in an age where knowledge is increasingly being integrated in emerging domains such as mathematical genetics; bio-informatics; nanotechnology; modeling; ethics in genetics and medicine; ecology and economics in the age of globalization, the curriculum in most parts of the world is typically administered in discrete packages. The analogy of mice in a maze appropriately characterizes a day in the life of students, with mutually exclusive class periods for math, science, literature, languages, social studies etc. Even mathematics is increasingly viewed as a highly specialized field in spite of its intricate connections to the arts and sciences. The thinkers of the Renaissance did not view themselves simply as mathematicians, or inventors or painters, or philosophers or political theorists, but thought of themselves as seekers of Knowledge, Truth and Beauty. In other words there was a Gestalt world-view with polymaths that worked back and forth between multiple domains. The results of this three year study with 120 pre-service students indicate that nearly half of the students displayed polymathic traits—as a result of their attempt to unravel the given paradox. This suggests that interdisciplinary activities can certainly play an important role in the education of these future teachers. By taking a phenomenological approach and trying to understand the first person perspective of these students, deep set hidden beliefs about the nature of mathematics also became apparent through the voices of the students. The pre-service teachers also reported an increased interest in the place of paradoxes in mathematics, which they had believed as an infallible or absolutist science.

However the significance and applicability of these findings come with certain limitations. It should be noted that nearly one-fourth of the students (group 3) in the study experienced and reported negative affective experiences as a result of tackling the paradox, which need to be sensitively attended to by the teacher educator. Debriefing sessions conducted during the course of the study were essential to create a positive pedagogical atmosphere on these students and foster a willingness on their part to try

Table 2 Polymathic themes emerging from journal writing and interview transcript analysis

Student groups	Polymathic traits					
	Shifts in perspective (paradoxes in art and language)	Thinking in analogies	Nepistemological awarness (mathematics as fallible)	Interest in investigating paradoxes	Mathematics as language (need for discourse)	Mathematics as philosophy
Group 1 (52) (unraveled paradox)	16	25	38	52	18	28
Interview Subgroup 1 (10)	6	10	6	10	5	4
Group 2 (40) (saw no contradiction-claimed the barber shaved himself)	7	12	4	4	2	4
Interview subgroup 2 (6)	3	2	1	3	2	3
Group 3 (28) (gave up on the problem)	1	5	1	0	1	0
Interview subgroup 3 (4)	0	3	0	0	1	0

interdisciplinary activities in their classroom which integrates mathematics with other subjects. Another missing ingredient in this study voiced by many students in their journal writings and interviews was the need to engage in discourse with other students when confronted with the paradox. It would be of interest to the community of researchers to investigate how pre-service elementary teachers tackle paradoxes in a collaborative group effort. Due to the limitations in the resources available for this study, the author did not pursue this approach but this remains a fertile area for further investigation. Finally, many of these students voiced concern over concrete ways in which interdisciplinary activities could be introduced in the elementary classroom. To this end there have been recent attempts to classify works of mathematics fiction suitable for use by K-12 teachers in conjunction with science and humanities teachers to broaden student learning. Padula (2005) argues that although good elementary teachers have historically known the value of mathematical fiction, mainly picture books, through which children could be engaged in mathematical learning, such an approach also has considerable value at the secondary level. Padula (2005) provides a small classification of books appropriate for use at the middle and high school levels, which integrate paradoxes, art, history, literature and science to "stimulate the interest of reluctant mathematics learners, reinforce the motivation of the student who is already intrigued by mathematics, introduce topics, supply interesting applications, and provide mathematical ideas in a literary and at times, highly visual context" (p. 13).

Mathematical paradoxes played a significant role in the historical development of the field. These paradoxes contain enormous pedagogical potential for pre-service teacher education, particularly in showing that even mathematics can be fallible. As seen in this study, realizing the fallibility within what students believed was an absolutistic and monolithic structure, these prospective teachers experienced both a sense of empowerment and expressed changing views about the nature of mathematics. It is the authors hope that this line of research is further developed by the community of researchers and teacher educators to make a positive impact on the beliefs and practices of future teachers of mathematics.

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