An Ode to Imre Lakatos: Quasi-Thought Experiments to Bridge the Ideal and Actual Mathematics Classrooms

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ABSTRACT: This paper explores the wide range of mathematics content and processes that arise in the secondary classroom via the use of unusual counting problems. A universal pedagogical goal of mathematics teachers is to convey a sense of unity among seemingly diverse topics within mathematics. Such a goal can be accomplished if we could conduct classroom discourse that conveys the Lakatosian (thought-experimental) view of mathematics as that of continual conjecture-proof-refutation which involves rich mathematizing experiences. I present a pathway towards this pedagogical goal by presenting student insights into an unusual counting problem and by using these outcomes to construct ideal mathematical possibilities (content and process) for discourse. In particular, I re-construct the quasi-empirical approaches of six14-year old students' attempts to solve this unusual counting problem and present the possibilities for mathematizing during classroom discourse in the imaginative spirit of Imre Lakatos. The pedagogical implications for the teaching and learning of mathematics in the secondary classroom and in mathematics teacher education are discussed.

KEYWORDS: Combinatorics, conjecture, counting, generalization, Lakatos, mathematization, mathematical structures, pedagogy, problem solving, refutation, teacher education.

Introduction

Imre Lakatos' (1976) classic book *Proofs and Refutations* contains an imaginary account of classroom discourse between the students and the teacher in the ideal classroom. This rich discourse occurs in the historical context of the problem of classifying regular polyhedra and constructing a proof for the relationship between the vertices, faces, and edges of a regular polyhedra given by Leonhard Euler as $V + F - E = 2$. The various protagonists of the book and the teacher passionately argue over the validity of definitions, explore and conjecture plausible pathways to a proof, and refute the validity of definitions and steps in a proof by producing pathological (monstrous) counterexamples. The discourse one finds in this book occurs in Lakatos' rich imagination but begs the question as to whether such a discourse is replicable in the classroom. Yet, nearly 30 years have passed without any practical follow up to Lakatos' vision for discourse in the ideal classroom. In this paper, the teacher attempts to re-construct the Lakatosian classroom by reflecting on pathways student discourse could take based on their responses to an unusual counting problem. The essence of the Lakatosian method lies in paying attention to the casting out of mathematical pathologies in the pursuit of truth. Typically one starts with a rule and clearly identifies the hypothesis. This is followed by an exploration of the possibility of its truth or falsity. The process of conjecture-proof-refutation results in the refinement of the hypothesis in the pursuit of truth in addition to the pursuit of all tangential hypotheses that arise during the course of discourse.

There are instances of mathematics educators creating a classroom environment that is conducive to such discourse. An outstanding example is the two-year study conducted by Harold Fawcett in the 1930s. Fawcett (1938) was successful in structuring a two-year teaching experiment with high school students that highlighted the role of argumentation in choosing definitions and axioms and illustrated the pedagogical value of working with a "limited tool kit." The students in Fawcett's study created suitable definitions, choose relevant axioms when necessary, and created Euclidean geometry by using the available mathematics of Euclid's time period. As a result, in this two-year teaching experiment each student in the classroom essentially created their own version of Euclidean geometry by choosing and defending their definitions, axioms, theorems, and proofs. The glimpses of the discourse one finds in Fawcett illustrates that the Lakatosian vision of the ideal classroom can in fact be approximated in reality. Fawcett's study is often quoted in math-education papers that expound the value of teaching Euclidean Geometry or Fawcett's instructional approach that allowed for Euclidean Geometry to be created as opposed to learned from a textbook. Other good recent examples of classroom pedagogy that emphasized the value of classical construction tools and the role of argumentation in geometry are recent classroom studies in the Netherlands. In these studies school children utilized instruments such as the conic section drawer (Van Maanen, 1992), while others performed classic geometric constructions following instructions on ancient manuscripts, for example, a pentagon from an ancient Persian manuscript (Hogendijk, 1996). My observation is that the crucial element in all the successful projects in the study of geometry reported was the classroom teacher, who had the pedagogical skills and a deep interest in historical content. Other examples are the use of historic clubs in which students act-out or role-play or study a historic text such as the *Elements* (Brodkey, 1996) and engage in argumentation whist establishing the truth of a given proposition (Fawcett, 1938).

The Lakatosian vision of classroom discourse and Fawcett's (1938) exemplary example (and others quoted above) suggests that *mathematization* is indeed possible in the secondary classroom. By mathematizing, I mean the "act of putting a structure onto a structure" (Wheeler, 2001, p. 51). A specific process that mathematization encompasses in the act of generalization where one imposes / discovers a generalization underlying varying problem situations (Sriraman, 2004a, 2004b, Sriraman & Adrian, 2004a). Wheeler (2001) pointed out the difficulty of teaching teachers to analyze and understand students' ability to mathematize, and the urgency to engage their students in mathematizing real problems by facilitating discourse. By real problems, Wheeler did not mean problems situated in a real-world context that can often be contrived, but problems that lead to non-trivial and tractable mathematics.

The Use of Atypical Counting Problems

Counting can be regarded as an activity that distinguishes us from other species. The history of mankind can be traced through the primitive acts of sharing equal quantities onto to the development of numerals that abstract the physical act of counting, onto to the development of sophisticated place value number systems that furthered our progress onto modernity. The history of mathematics is peppered with numerous famous counting problems. Galileo puzzled over the cardinalities of the set of integers and the set of even integers. This problem eventually led to the construction of transfinite arithmetic by Cantor as a reconstruction of ordinary arithmetic (Rotman, 1977). One can boldly generalize and say, "Everyone can count!" (to an extent), which begs the question of why counting problems occur less and less frequently as students' progress onto the secondary grades. Curricular documents from organizations like the Australian Education Council (1990) and the National Council of Teachers of Mathematics (2000) are rather vague about the place of counting problems in the secondary mathematics curriculum. By *counting problems* I mean situations where a certain phenomenon is observed among numbers (usually positive integers), which naturally lead one to examine the reasons why such a phenomenon occurs or problems that require a basic number theoretic insight such as divisibility of a number, properties of remainders, and so forth. These types of problems are sometimes stressed under the discrete math curricular strand, which encompasses basic topics from combinatorics. However the point I will try to stress in this paper is that one does not need a text or a curricular requirement in order to make use of counting problems in the classroom. In particular, my intent is two-fold:

1) To demonstrate the wide range of mathematics that become accessible in the secondary classroom via use of counting problems.
2) To present the possibilities for mathematizing via student insights into the problem.

Pedagogical Hopes / Goals and its Reflections in the History of Mathematics

One of our universal curricular goals is to convey a sense of unity among seemingly diverse topics within mathematics. In order to show a pathway into achieving these aforementioned goals I present the outcomes of the use of one such novel problem and the ensuing mathematical possibilities resulting from student insights into the problem. These students are prototypical of above average high school students (13-15 year range) at a rural American high school enrolled in Accelerated Algebra I, a course for motivated students. Over the course of the school year, as the teacher of this class, I assigned problems that students solved in their journals. The pedagogical hope was to mediate conditions that allowed students to investigate open-ended problems over an extended time period, to encourage students to write out their solutions in depth, and to get them to reflect on their solutions. Students worked on these problems in their journals for seven to ten days and were interviewed subsequently. Another pedagogical hope was that students would be inspired to adopt a quasi-empirical methodology when tackling problems they had never encountered before in their schooling experiences.

This hope was inspired partly by following the lead of how instruction is typically tailored in science. Science is characterized by physical principles discovered or inferred via systematic observation, hypothesis generation, and testing through experiment. For instance, in a high school science lab experiment the validity of a scientific principle is tested by performing a structured experiment, recording observations followed by the application of the appropriate regression techniques (or other means of data analysis) on the data to test the validity of the principle. Some innovative teachers would set up a structured science experiment, in which students gathered data and then tried to infer the principle that worked. Can this scientific method be somehow adapted to the learning of mathematics? Can teachers facilitate the discovery of mathematical generalizations and underlying principles by using structured problem situations that result in the adoption of a quasi-empirical methodology characterized by construction of particular cases and observations and eventually result in new mathematics? Can students, when perplexed with a problem unsolvable with their extant mathematical toolkits or repertoire, be pushed to create new mathematical tools required to solve a problem? As the old adage says "necessity is the mother of invention" and the history of mathematics is characterized by this necessity for building new tools to tackle troubling problems. Besides Cantor's creation of transfinite arithmetic mentioned earlier, there are other historical examples that illustrate my point. For instance, Goldbach's (1742) conjecture that all even integers $ 4 can be expressed as the sum of two primes as yet remains unanswered, but as a consequence has resulted in the search and the creation of new mathematical machinery, both computational and theoretical to tackle this problem.

Finally I cannot help but point out the phenomenal growth of mathematics and the many beautiful theorems of the 20th century as a consequence of the willingness of most mathematicians to use the Axiom of choice.[1] Barry Lewis, the 2003-2003 president of the Mathematical Association in the United Kingdom recently gave a broad perspective on the value of tool building in mathematics.

> It was no accident that when at the beginning of the last century Einstein needed different tools to look afresh at the space-time continuum, those tools had already been fashioned … in an abstract system with no possible practical use. At least that is what Gauss thought for he didn't even bother to publish his work. (Lewis, 2003, p. 426)

In summary, can we use the inherent structure and beauty of a mathematics problem to inspire an exploration of mathematics through student insights into the problem and actually conduct classroom discourse in the spirit of Lakatos' (1976) exemplary and imaginative discourse in the ideal classroom? Let us see.

The Problem

Consider the following problem (Gardner, 1997): Choose a set S of ten positive integers smaller than 100. For example I choose the set S = {3, 9, 14, 21, 26, 35, 42, 59, 63, 76}. There are two completely different selections from S that have the same sum. For example, in my set S, I can first select 14, 63, and then select 35, 42. Notice that they both add up to 77 (14 + 63 = 77; 35+ 42 = 77). I could also first select 3,9, 14 and then select 26. Notice that they both add up to 26 (3+9+14 = 26; and 26 = 26). No matter how you choose a set of ten positive integers smaller than 100, there will always be two completely different selections that have the same sum. Make up sets of your own and check for yourselves. Why does this happen? Prove that this will always happen.

Student Pathways, Insights, and Tool Building

The problem is clearly intriguing and conveys a sense of the mystery of the integers. I encouraged students to randomly pick integers between 1 and 100 and we constructed several ten-element sets. I now give an account of and re-construct the quasi-empirical approaches of six 13-15 year old students as they attempted to solve the aforementioned counting problem in their journals. Student journal writings and interview vignettes are used to re-create student pathways and insights into this problem.

Devising Techniques to Control Problem Variability

One of the students in the class (Matt) was particularly adept at discovering different selections of numbers that yielded invariant sums. However he did not believe that this always occurred and embarked on constructing a counterexample based on controlling how the digits were chosen in the problem. Matt first tried to control the variability of the problem by holding the number in the tens place constant and then varying the numbers in the units place and conjectured that since one is being forced to repeat one or more of the digits from 0 through 9 in picking the ten elements this results in two sums being the same. For instance one can set the tens place as 8, and then start picking the digits for the units place, and we have ten choices for the digit in the units place (namely the digits 0-9) but then in the process one is forced to repeat the digit 8 (since the ten integers have to be different). Matt's original conjecture was actually based on the set {90, 91, 92, …, 99} where one can select distinct digits until one has to select 99 where the 9 repeats. This according to him caused different two selections in a ten-element set to yield the same sum. He then conjectured that this was not the case if one picked a set with less than ten numbers and produced the 5 element set {3, 7, 12, 78, 69, 84}, with the number 78 crossed out, and a list of the sums: 3+7=10; 3+12= 15; 3+69= 72; 3+84=87; 7+12=19; 7+69=76; 7+84=91; 12+69=81; 12+84=96; and 69+84= 153. Matt had crossed out the 78 because he had accidentally repeated the digit 7 and his scheme was to construct a set in which all the digits were unique. He concluded that this set illustrated his point of different selections yielding the same sum only when one repeated the digits. Matt constructed a five-element maximal set in the digits from 0 to 9 occurred only once in order to illustrate his scheme but he did not consider sums, which combined three or more numbers in his set. If he had done so, he would have noticed that 3+12+69 = 84, an element in his set, thus disproving his claim (Sriraman, 2004a).

This leads one to wonder whether students had unconsciously understood the requirement of different selections yielding the same sum as strictly sums of two numbers. Consider the following solution of John.

Discovering the Counting Rule for Subsets of a Set

John's plan was to take ten numbers and see if this worked, that is, if two different two sums that equaled each other could always be found. He was going to try this numerous times to "see that the sum always equals another number." So, John's plan was to model the problem and verify if this was true, as in the first two problems. John started by picking the set {1, 15, 13, 4, 6, 99, 20, 75, 86, 51}, and found the sum 13+86=99, which is an element of the set. He then found that 75+6+4+1=86, which is another element in the set. I thought that John's second sum was really elaborate and made a note to ask him if he had developed a certain method to construct the sums. The next set picked by John was {2, 10, 18, 25, 52, 49, 91, 1, 86, 98}, and he found the sum 86+10+2=98, an element of the set. Clearly John understood different selections the way it was intended in the fact that he picked three and four digit sums.

The third set picked was {90, 91, 92, 93, 94, 95, 96, 97, 98, 99}, which I thought was a very interesting set to pick, because it gave the largest sum possible for sets containing ten integers between 1 and a 100. This fact will come into play when we discuss the solution later. The two sums found by John in this set were, 91+92= 183 and 90+93=183. In his summary of the problem, John wrote that he had solved the problem exactly as he had planned. "I took sets of ten three times, and it worked every time." He then wrote that he was unsure about why this happened but his guess was that "there were barely enough numbers for it to work out."

John was asked to clarify his thought processes during the interview but had to cancel his first appointment because of a tennis tournament. When he came to the interview his journal had a *second look* to the problem. Apparently John had decided to look the problem over again. In his second look, John constructed the set {2, 5, 6, 8, 10, 12, 14, 16, 19, 20}. He found the sums 6+8 = 2+12 =14; 2+10=12, an element in the set; 8+2=10, an element in the set; and 6+2 = 8, another element in the set. John then picked a second set {1, 3, 5, 7, 11,13, 15, 17, 19}, and wrote "5+3+1 =9, and 9+7+1=17 and so forth." I noticed that John had first picked integers at random between 1 and 20, and then picked a set of odd integers between 1 and 20. He was clearly beginning to choose the integers carefully at this stage, probably with the hope of finding something. In the margin of his journal he added the numbers 1+2+3+4+5+6+7+8+9+10=45 and he drew the following chart.

1	2, 3, 4, 5, 6, 7, 8, 9, ~~10~~
2	3, 4, 5, 6, 7, 8, ~~9, 10~~
3	4, 5, 6, 7, ~~8, 9, 10~~
4	5, 6, ~~7, 8, 9, 10~~
5	6, 7, 8, 9, 10
6	7, 8, 9, 10
7	8, 9, 10
8	9, 10
9	10
10	

Figure 1. *John's representation of the Number Sum Problem.*

Based on his sum and the chart (Figure 1) John concluded, "I am thinking that it always happens because there are numerous amount of ways to get the answer. I made a chart and just adding two numbers, there are 45 ways. If you add three numbers there is an even greater chance." I was at first perplexed by John's solution, and tried to make sense of John's concluding argument. John was essentially saying that one could calculate a very large number of possible sums, which was a very good observation. The reader should note that these students had not encountered notions of enumerating subsets of a given set, which plays a role in the solution. However John was beginning to devise a systematic way of counting all such subsets. In his chart, John had calculated the total number of two number sums in a set containing ten elements, which is 45, or ten choose two in combinatorial terms. He then realized that there were even more possibilities for sums containing three numbers, and based on this observation concluded that this phenomenon occurred because "there were numerous amount of ways to get the answer." John had generalized that a large number of sums were possible, by counting all possible two number sums, and then by noting that an even greater number of three number sums were possible. This scheme highlights the possibility of students independently constructing the rule that the number of elements of a k-element set is 2^k.

John did not play with the given starting conditions of the problem. In other words he only tried 10 element sets, unlike Matt who decided to play with the constraints and see what happened. There were others who followed a different line of experimentation, which is now revealed in the following attempt of Jim.

Pondering Over the Mystery of the Number System

Jim understood the problem "to be saying that out of any 10 number set of positive integers less than 100, you can pick 2 numbers to add up to a number in the set. You could also take 3 numbers … there is also the possibility of 2 numbers adding to a number on both sides." His plan to start the problem was to "randomly make a 10 number set, and find at least one combo, then make another one to make sure it always works." Jim's plan basically involved verification of the already stated fact in the problem. He constructed two sets which were {1,33,44,71,52,27,13,69,88,97} and {3,14,23,39,40,55,67,71,83,99} and found the sums 49+52=13+88=101, and 23+14+3=40 respectively. There seemed to be no obvious system that Jim was using to find sums that were equal. Jim did not make up any other sets but concluded, "the answer deals with the base 10 system. There are really only 10 numbers to choose from, anything higher can be represented as 10+x, its not like there is a new symbol like ~ to stand for something. For example 11=10+1, and 52=10+10+10+10+10+2. "He then wrote that every number between 1 and a 100 would be a "combo" (addition) of 10 and another digit from zero through nine.

I did not understand Jim's "combo" argument. It seemed that the operation of addition on the base ten symbols somehow explained to Jim why the sums worked. I hoped that the interview would yield some insights into what Jim was trying to say. This particular vignette has been condensed for the reader. To the reader that wished to skip it, the essence of Jim's argument was as follows. "It worked because the only symbols that were available came from the set {1,2,3,4,5,6,7,8,9,10}" and all numbers between 1 and a 100 were some additive "combination" of these symbols, and this somehow ensured that two selections added up to the same sum. To the reader that would like to see how Jim came to the above conclusion, the vignette follows:

Vignette 1

Student: It took me a while to think abut the base 10 system thing. I had a couple of number lines, like ten number sets, and I always found numbers to add up to the same number. Then, I like, *on my paper route was thinking about it, and then it suddenly occurred to me that there were only ten numbers to choose from.* Everything else can be expressed as 10 plus something. Like 11= 10 + 1, and 50 = 10+ 10+ 10+10+10, and it is not like there are any new symbols, *like 11 doesn't have a new symbol, it is just 10 and 1.*

Author: Okay, so you are saying that the only numbers to choose from are?

Student: 1,2,3,4 … 10, all the other numbers are a variation of that form.

Author: Okay, but how does that make two sums to be the same?

Student: Because these are in order, and they are smaller than 50, so it is easy to see them.

Author: What if you were to pick numbers at random? Like the ones I showed you in class?

Student: Then *a good idea would be to organize them, after you have randomly picked them, from least to greatest, and then it is easy to add up,* instead of adding by jumping around and stuff, so then you find as many possibilities as you can. *Like you take the first number and go along the line see if it will add up to any number in the set. Then you go to the next number, and then add up two.*

Author: So how many days did you spend on this problem?

Student: Like thinking it out, maybe 5 days, and then I wrote up task one or two days ago, then I wrote everything up last night, and that took about me about 45 minutes.

Author: So you thought about it for 5 days?

Student: Yeah, I've been thinking about it for a long time (laughing).

[*The interview was resumed the following day as Jim had to leave for an after school activity*]

Next Day

Author: Okay, here we are again.
Student: I somewhat wrote down why I think this happens. I think its' because of the set 1 through 10 and stuff, because the only numbers you can get are, you choose from that set.
Author: How does picking ten numbers at random relate to your ten number set with 1 through 10?
Student: Cause this, say 88 is just a combination of two of these numbers or something. Like any number you can have is one of these or a combination of these. So everything you make has to be a combination of these.
(We are looking at 49+52=13+88)
Author: Okay say, I take 49 from your set, what do you mean by a combination?
Student: Take 4 and 9 and put them together and you have 49.
Author: And the 52?
Student: That's 5 and 2.
Author: Now how does that relate to 13 and 88?
Student: Cause they come from the same set, its not like a different thing you pull the numbers from. It's from the same 1 through 10 set.
Author: I'm still not sure about what you are choosing?
Student: You pick like the digits, you can get a number for each digit out of this set.
Author: But how does that make two sums equal?
Student: Like all the numbers in here can add up to each other, like 1+1 =2, 1+3 =4, and you can on like that.
Author: Okay. But how does that make 49+52=13+88? How do you go from there to there?
Student: This is just a more advanced way or something. Instead of just one digit, it has two.

Jim was trying very hard to explain his solution. He was suggesting that the numbers in a ten number set would involve some combination of the digits from zero through nine, but it was still unclear to me how this would lead to two sums being equal. He attributed it to the base 10 system and how things add up. The reader may recall that Matt presented a similar argument which involved the digits from zero through nine, but in his case he tried to verify his conjecture by trying sets with less than ten numbers and by controlling the choice of digits in these sets. In Jim's case, he was trying to gain an insight into the problem by reflecting over the sums in the set {1,2,3,4,5,6,7,8,9,10} and observing that all sums could be expressed in the form 10a+b, where a and b 0{0,1,2,3,4,5,6,7,8,9}. This led him to conclude that two selections yielded the same sum because of how addition worked on the finite number symbols. Unlike John and Jim, Jamie decided to experiment with the starting conditions of the problem in order to gain an insight into why these mysterious sums occurred.

Playing With the Given Hypothesis

In her journal, Jamie began the problem by writing, "the problem is asking you to first have a set of ten positive numbers all smaller than 100. The problem states that there will always be two selections from S that will have the same sum. It does not matter what the numbers are. What I need to figure out is why this happens and give examples why it does." The reader will note that in restating the problem, Jamie had translated proving to "give examples why it does."

In order to start the problem, Jamie decided to pick out ten integers for her different sets of numbers. Then she would find selections that gave the same sum. She wrote, "I may have to try different additions before I find sets that work, but there will always be two different selections that will have the same sum."

Jamie tried three different sets, and in each case found two selections that gave the same sum. Jamie's first set was {1, 2, 16, 19, 25, 40, 45, 72, 75, 89} which gave 1+2+16= 19, an element in the set. She also found 72+16+1= 89 an element in the set. Her second set was {3, 12, 15, 30, 35, 50, 63, 74, 87, 99} which produced 3+12=15, an element in the set, and 87+12=99 an element in the set. Her final set was {89, 90, 91, 92, 93, 94, 95, 96, 97, 98} which yielded 90+92=89+93=182 and 93+95=94+96=188.

At this point Jamie concluded, "I don't know how to prove this will always happen. There is probably some kind of theory why it does though. I don't know what this theory is or how to figure out what it is. I do think this always happens because it may have something to do that we are using only 10 integers." The reader will note that her selection of the ten integers became less random in her third set, where she chose consecutive integers from 89 through 98. She had not devised a system for finding equal sums. The author hoped that the interview would reveal why she had not pursued the problem any further.

Vignette 2

Student: I tried out different sets and it worked every time. I really couldn't figure out why it worked, a theory or anything.

Author: So you could always find two selections?

Student: Yeah, I tried out different sets, and it always worked. I don't really know why it worked? I couldn't really prove a theory or anything (sighing).

Author: So, you tried different sets of ten numbers each.

Student: (Silence) I tried to figure out why it works?

Author: What did you think about?

Student: I thought maybe it has to do with like, that it can't be negative numbers and it has to be between one and a hundred.

A. Several sets with ten negative integers between –1 and –100.
B. Several sets of varying size, containing positive and negative integers between 1 and 100 and –1 and –100.
C. Several sets with fifteen positive integers between 1 and 200.
D. Several sets of varying size, containing positive integers between 1 and 100.

Note: In each of the four categories, Jamie tried to construct an argument to necessitate the acceptance of the hypothesis in the given problem. The arguments constructed by Jamie in her journal for each category were:

A. {-1, -13, -65, -72, -73, -86, -89, -90, -96, -99}
-89 –1 = - 90 ; -73 –13 = -86.
If the rules were to use ten negative numbers less than 100 it would also work. But this would not solve the (given) problem because we have to use positive numbers.

B. { 3,7,25,31} ; { -17, 27, 52}
Negative and positive numbers are used. These sets are less than 10 (elements). Nothing works here!

C. {3, 17, 24, 74, 84, 91, 93, 96, 108, 14, 121, 135, 145, 157, 163}
93+ 3 = 96 ; 91 + 17 = 108 ; 121 + 24 = 145.
It could still work if some of the guidelines are further changed. Instead of numbers less than 100 it could be numbers less than 200 and fifteen numbers instead of 10.

D. { 1,3}
Cannot work with two numbers because there is no way to get two selections that work.
{1,2,3,4,5}
It could work here since 1+2 =3 and 1+3 = 4. But what if we had {72, 93, 94, 95, 96} it does not work.

Figure 2. *Jamie's experimentation with the hypothesis in the problem.*

Jamie said that she would look at the problem again and then talk about it. Jamie's second attempt involved construction of numerous sets. This time Jamie had modified the constraints of the problem as shown in Figure 2.

Based on her experimentation with the given hypothesis in categories A, B, C, and D (Figure 2) Jamie concluded. "In order to always get the solutions that work, there has to be 10 or more integers that are positive and I don't think it matters what the actual numbers are, like it does not have to be strictly under 100, it could be more." Jamie used examples/counterexamples in cases where the given hypothesis was tweaked in order to justify the correctness of the given hypothesis. This is certainly an interesting approach. It was also noteworthy that she attempted the problem again and tried to vary the given constraints in order to get some insight. The next solution illustrates the construction of one particular counter-example for the problem where the hypothesis was changed to sums of a 4– element set and shows similarities to Jamie's line of thought.

Constructing a Particular "Pathological" Case

Hanna began the problem by rewriting the example given by the researcher/teacher. She then posed to herself the question, "why does this happen in every set of ten positive integers smaller than a hundred?" In order to answer this question, Hanna decided to "start by choosing a couple of sets of ten positive integers less than 100 and see if two different selections equal the same." After this, she would "look at her results and try to understand why?"

The first set that Hanna made was {19, 7, 29, 30, 3, 21, 15, 16, 17, 28} and she found the sum 16+3=19 an element of the set. Her second set was {1,2,3,4,5,6,7,8,9,10} and she found the sum 2+3=5 an element in the set. Her third set was {2,3,5,8,4,6,7,1,10,11} where she found the sum 5+4=8+1=9. The fourth set was {11,3,19,7,30,2,4,6,5,9} and she found the sum 19+6+5=30, an element in the set. She tried four more sets, each with ten elements, and always found two selections that gave the same sum.

I could not detect any refinement in how Hanna chose the integers. The first six sets that Hanna constructed had integers that were less than 20. She had one set that had the integers from 1 to 10. She had also not devised a systematic way of calculating all the possible selections that would yield two equal sums. Her conclusion as to why this phenomenon was occurring based on her work on the eight sets was as follows:

> Out of all the sets of ten positive integers that I picked, they all had two selections that added up to equal the same number. I think this might happen because you aren't using any negative numbers and because you are not subtracting, so you won't have negative numbers. I think this might happen because you can pick ten numbers under 100, so that gives you a wide variety to pick from. Also ten numbers gives you quite a few numbers to play with and add to give you equal sums .... But if you could only have a set of 4 or 5 numbers under a 100, you wouldn't always get 2 equal sums because you have less numbers to deal with. (Journal Entry)

Hanna then constructed a four element set {19,2,48,1} and calculated all the possible sums in this set besides the trivial sums of 19; 2; 48, and 1. The remaining sums were; 19 + 2 = 31 ; 19+ 48 = 67 ; 19 + 1 = 20 ; 2 + 48 = 50 ; 2 + 1 = 3 ; 48 + 1 =49; 19 + 2 + 48 = 69 ; 19 + 2 + 1 = 22; 19 + 48 + 1 = 68 ; 2 + 48 + 1 = 51 ; 19 + 2 + 48 + 1 = 70 and concluded that "nothing in that set equals the same because there isn't enough numbers that you could put together to equal a certain sum that would equal other selections sum."

Her argument that there was a great deal of variety (in terms of possible selections) in a ten number set was plausible and similar to that of John's. She had conjectured that a smaller set would not have variety and hence two selections would not give the same sum in such a set. She supported her conjecture by constructing a four-element set in which two selections did not yield the same sum. This counterexample would fall into category 2 of Jamie's experimentation. Having seen four approaches to the given problem, I finally present one very unusual approach devised by Amy.

Constructing an unusual set

Amy was intrigued by this problem and constructed an unusual set in the process of experimenting with various number sums. After constructing numerous ten-element sets, Amy was certain of the phenomenon of different selections yielding the same sum. She wrote in her journal that repeated verification of this phenomenon simply validated that it always worked and in order to find out why different selections yielded the same sum, a completely new approach was required in which each element was purposefully chosen. She constructed the set { 1, 2, 4, 8, 16, 32, 64 …} and wrote

> Ok, I think maybe I got somewhere! Ok, to choose this set I tried to make the most variety possible. Let me try to explain … I started out with 1, then I chose 2. Now, I didn't want to get a solution, so my next number obviously wasn't going to be 3, so I put 4 instead, because 1+2=3 and I would have had a solution already. So, I just continued working this way. The next bigger number would be 7, because 1+2+4=7, so I didn't choose 7, instead I chose the next number 8. Now, 1 + 2 + 4 + 8 =15, so I wouldn't want to pick 15, because that would be a solution, so I chose 16. When I had these five numbers {1, 2, 4, 8, 16} … I discovered a pattern. I just had to keep doubling the last number to get the next number … (and got) the set {1, 2, 4, 8, 16, 32, 64, …}. (Sriraman, 2003a)

Once this set was constructed, Amy wrote that she couldn't continue with her scheme since the problem required that the elements in the set be between 1 and 100. She conjectured that her 7-element set was a maximal set, which did not yield selections of numbers that added to the same sum (Sriraman, 2004b).

The Lakatosian Possibilities of Discourse by Mathematizing Student Strategies / Solutions

Having presented six student pathways into this problem, we are now in a position to reflect on the nature of the outcomes and appreciate the mathematical possibilities (content and process) in the classroom. In this section I first present plausible classroom scenarios based on the actual student insights discussed in the previous section. In doing so I am putting myself in the shoes of the reflective teacher that analyzes student insights to recognize the rich mathematizing experiences in these student insights with the goal of facilitating classroom discourse that lead to non-trivial mathematics. One can indeed imagine the Lakatosian setting where these six students are present in the ideal classroom, engaged in classroom discourse based on their attempts of the given problem, where the teacher merely facilitates the mathematizing experiences that lead to non-trivial mathematics.

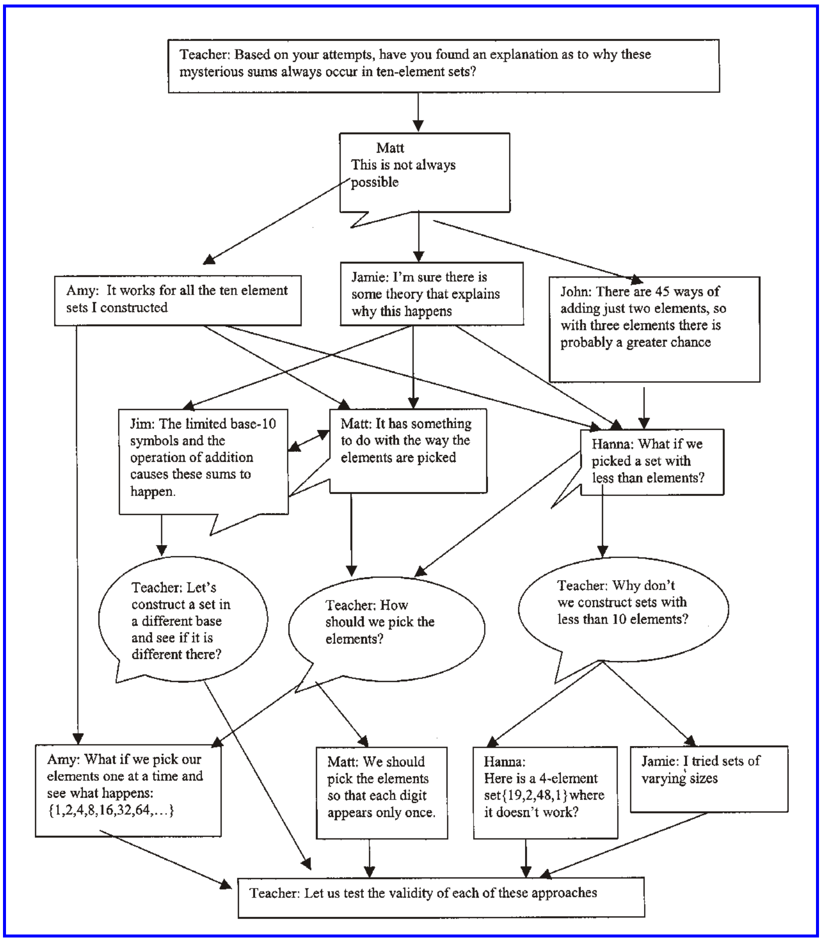

Figure 3. *Plausible discourse based on student solutions and strategies.*

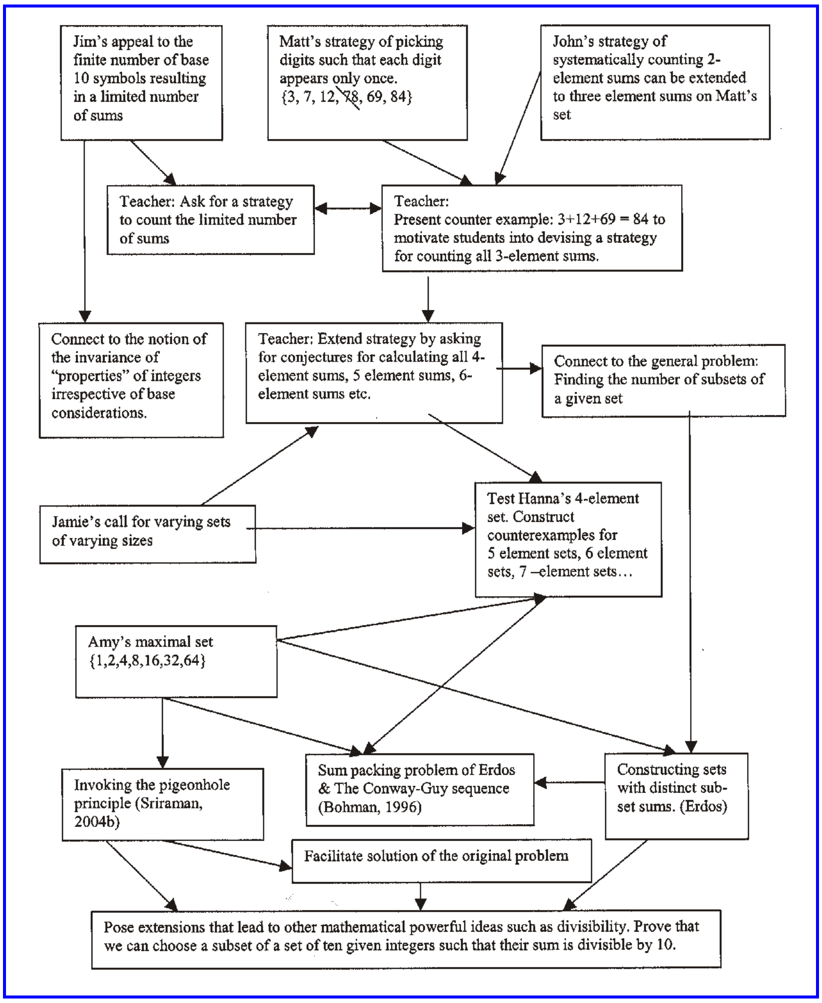

Figure 4. *Pathways into mathematizing experiences via conjecture-proof-refutation in discourse.*

Figure 3 shows plausible classroom discourse based on student strategies and insights. Figure 4 constructs possible pathways into diverse content based by mathematizing student strategies. This figure also contains elements of the Lakatosian process of conjecture-proof-refutation to lead to rich mathematics. In the next section both figures are discussed in depth in addition to a discussion of the actual mathematics arising from mathematizing student insights / strategies to fruition, that is, discovery of general structures and deeper results.

Discussion of Plausible Discourse Pathways

The discourse begins in Figure 3 with the teacher asking the six students whether they were able to find a reasonable explanation for the mysterious phenomenon of different selections yielding the same sum in the given problem. Matt, who was especially an enthusiastic student in this course, does not agree this is possible based on his strategy of picking elements so that no digit repeats. Amy, Jamie, and John who have verified this phenomenon for numerous 10-element sets point out that these sums occurred in all the 10-element sets they have constructed. Jamie thinks there is some theory that explains why this happens. John suggests that these sums occur because of the number of ways in which one can calculate such sums in any 10-element set, and says that there are 45 ways of combining two elements at a time. At this juncture there are several pathways that the discourse can take. Conceivably, Matt points out that these sums occur because of a particular method of choosing the ten elements. Hanna suggests that such sums would not occur if one picked a set with less than ten elements. Jim builds off Jamie and Matt's remarks and believes the sums to be a function of the finite way in which one can choose the ten elements using the digits of the base-10 numeral system. At this juncture the teacher can pose several questions that will facilitate / encourage the students to share their examples with the class. For instance asking "How should we pick the elements" would allow Amy and Matt to share their insights of constructing a seven-element maximal set where no sums occur, and constructing a set in which no digit is allowed to repeat respectively. The teacher could also ask for examples of sets with less than ten elements, so that Hanna and Jamie can share their work with the others.

In order to further Jim's suggestion about the mysterious working of the base-10 system, the teacher can ask for sets of ten elements in a different base. The important thing at this stage would be to test the validity of all the conjectures / explanations by constructing the appropriate refutations and proofs in order to mathematize (and cash out!) the rich conjectures posed by the students. The teacher is also in a position to point out to the students their tendency of playing with the hypothesis is a natural practice among mathematicians in order to gain an insight into a given problem. The reader should note that although the discourse is constructed in the Lakatosian imaginative spirit, it is certainly very plausible since it is based on actual (factual!) responses of students in their journals and interview transcripts.

Discussion of Mathematizing Experiences Leading to Non-Trivial Mathematics

Figure 4 gives the rich possibilities of mathematizing student insights with the teacher facilitating (or orchestrating) this process. I also discuss the remarkable mathematics that arises from student insights into the problem. Matt's strategy of constructing a set in which each digit only occurs once {3, 7, 12, 78, 69, 84} in order to refute the suggestion that such sums always occur can be re-refuted by the teacher by presenting the counter example: 3+12+69 = 84 to motivate students into devising a strategy for counting all 3-element sums. This would allow John to share his strategy of systematically generating 2-element sums. Jim's appeal to the finite number of base 10 symbols resulting in a limited number of sums can also be now tested by asking students to estimate the number of sums possible in a ten-element set.

Students can be asked to conjecture a strategy for 3-element sums based on John's work and this can be extended indefinitely and nicely connected to the general problem of counting the number of subsets of any given set, thus illuminating the general structure of subsets of a set. The ambitious teacher can extend this further by asking students to construct sets with distinct subset sums, a problem posed by Paul Erdös in 1931, and pose the sum-packing problem of Erdös. The sum-packing problem of Erdös is as follows.

A set S of positive integers has distinct subset sums if the set { $^{3}{}_{x0X}$ x: X d S } has 2^{*s*} distinct elements. Let f(n)= min {max S: *S*=n and S has distinct subset sums}. How small is f(n)?

In 1931, Erdös conjectured that $f(n) \ \$ \ c\ 2^n$ for some constant c, and in 1955 Erdös and Moser proved that $f(n) \ \$ \ 2^n/(10\sqrt{n})$ and this remains as the best estimate for a lower bound (Bohman, 1996). Naturally we ask ourselves what is the upper bound? If we take the set S to be the first n powers of 2 as Amy did, we can easily see that $f(n) \ \$ \ 2^{n-1}$. Amy's unusual insight into the problem and her maximal set construction actually yield the first three numbers of the Conway-Guy sequence constructed by Conway and Guy in 1967 (Guy, 1982) arising from this non-trivial sum packing problem. The problem (Gardner, 1997) given to the students can be generalized to get the Conway-Guy numbers as follows. As defined above, we let f(n) be the smallest positive integer such that there exists n positive integers # f(n) for which all subset sums are distinct. The first three values of f are f(1) =1, f(2) = 2, f(3) = 4. Surprisingly enough f(4) is not 8 but 7. The bound of 8 is suggested by the fact that the subset sums of the first four powers of two -1,2,4, and 8 are clearly distinct. This was Amy's insight when she tried to construct a set with distinct sums (Sriraman, 2004b). Amy's idea might lead the reader (the "ideal" class) to believe that if four numbers # 7 are chosen, then two different selections from the set must have the same sum. This is shown false by considering the set S ={3,5,6,7} (Check it out!). Perhaps more surprising is the fact that f (6) = 24, not 32, which the reader might suspect by binary representation considerations. The corresponding set is S = {11, 17, 20, 22, 23, 24}. The first eight values of the Conway-Guy sequence f(n) are 1, 2, 4, 7, 13, 24, 44, 84 .The upper bound found by Conway and Guy on S was 2^{n-2} provided n is large enough. Such a pathway would mathematize and illuminate the grand structure beneath the original problem.

A different pathway to the sum-packing problem and the Conway Guy numbers is via the strategy for systematically generating subsets, and hence all possible sums can be used to test Hanna's four-element set. A natural question is whether we can construct five-element sets, six-element sets, seven element sets, and so on, with integers chosen from 1 to 100, such that different selections do not yield the same sum. This question would allow the teacher to take Amy and Jamie's ideas and mathematizate it to fruition by leading into the sum-packing problem. Finally, as mentioned earlier, Amy's seven element maximal set again naturally leads to the sum-packing problem of Erdös as well as the Conway-Guy numbers process with appropriate facilitation.

Another possibility for mathematization after students have constructed ways to generate subsets of sets and the number of such subsets is to have Amy invoke the pigeonhole principle (Sriraman, 2004b) in order to solve the original problem. The pigeonhole principle can then be used to facilitate the solution of the original problem by observing that the smallest possible sum is 1 and the largest possible sum is 90+91+…99 = 945, but the number of subsets of a ten element set is much larger than this, namely 2^{10}=1024, thereby forcing numerous selections of subsets to yield the same sum. This would merge the diverse ideas of the students from examining the set {90, 91,…99} (Matt) to efficiently calculating sums (John) to the hunch of the finite number of sums because of the limited choice of digits (Jim) to conforming the validity of the hypothesis for Hanna and Jamie. In fact now, Hanna, Jamie and the others could test the validity of their conjectures for the sets with less than 10 elements. This would confirm Jamie's hunch that a 15 element set with integers chosen from 1-200 would certainly yield numerous selections that add up to the same sum [$2^{15} > 190 + 191 + \ldots 199$]. There are a myriad of possible extensions that mathematize other powerful ideas such as divisibility and the wide-ranging applicability of the pigeonhole principle. For instance: Prove that we can choose a subset of a set of ten given integers such that their sum is divisible by 10 (Fomin, Genkin, & Itenberg, 1996).

Conclusion and Implications

The preceding discussion highlights the plethora of mathematics that becomes accessible to secondary students if the teacher makes use of student insights on atypical problems to lead students to discover mathematical structures. Although the six students only had the mathematical sophistication of beginning Algebra students and hence operating with a limited toolkit, their attempts revealed the natural tendency to create new mathematical tools to tackle this problem. In a sense their mathematical behavior was analogous to that of the rich tool building process characterizing the history of mathematics when mathematicians were confronted with perplexing problems such as Fermat's Last Theorem and The Four-Color Problem. Thus, creating mathematical experiences that necessitate the creating of new tools is a very useful pedagogical technique.

The value of choosing novel (atypical) problems that capture students' interests and seem accessible to students needs to be emphasized to practitioners. Numerous studies seem to indicate that combinatorial (e.g., English, 1998, 1999; Hung, 1998; Maher & Kiczek, 2000; Maher & Martino, 1996a, 1996b, 1997; Maher & Speiser, 1997; Sriraman, 2004a, 2004b, 2004c) and number theoretic problems (Sriraman, 2003b, 2004d ; Sriraman & Strzelecki, 2004a) are especially useful in serving the purpose of being accessible as well as lead into investigations of the underlying structure. The accessibility of simply stated (but mathematically complex) problems help teachers to foster independent thinking in the classroom (Sriraman & English, 2004a).

Although the current trend in mathematics education is to emphasize model-eliciting problems/activities (Lesh & Doerr, 2003) situated in a real-world context as a means to catalyze mathematizing in the classroom, novel pure math problems still have an important place in the curriculum. Encouraging students to tackle atypical counting problems to make deep connections with topics in Number

Theory, Combinatorics, and Analysis complements the Applied mathematics and Statistics that students learn through the modeling approach that is presently gathering momentum. The underlying hope is that mathematics educators never forget the aesthetic beauty inherent in pure math activities and convey to our students that such activities have sustained the imagination of mathematicians and contributed to its growth from the very onset of its history (Sriraman and Strzelecki, 2004a).

The value of allowing students extended time periods to work on problems and encourage them to engage in reflective journal writing serves multiple purposes. It not only creates a non-threatening medium via which students are willing to try multiple strategies, but it also helps the teacher to plan lessons that involve mathematical discourse with the goal of facilitating the creation of new mathematics as well as the discovery of structure. Student journal writings are also an invaluable asset to initiate and orchestrate classroom discourse with the aim of mathematizing in mind. Even incorrect student attempts or counter-examples serve the pedagogical purpose of allowing the teacher or other students to construct appropriate refutations that will allow students to make the necessary "tweak" to move the mathematics forward. The preceding student attempts on the given problem seem to indicate that students have a pre-disposition to experiment with the hypotheses when the problem is too difficult in the stated form. After all, new mathematics is created by this continual "tweaking" process in which preliminary hypothesis undergo refinement until a theorem emerges. The Lakatosian methodology of conjecture-proof-refutation conveys a vibrant and alive picture of mathematics. It would be worthwhile to expose prospective teachers by modeling this process in mathematics and mathematics education courses. The preceding hypothetical classroom discourse constructed with actual student insights reveals the rich mathematics that becomes accessible via the Lakatosian methodology and careful facilitation of discourse. This also implies the necessity to impress upon prospective teachers the value of an in-depth knowledge of graduate level mathematics.

Doerr and Lesh (2003) recently called for mathematics educators to recognize the applicability and the analogs of Dienes (1960, 1961) principles to teacher education. For instance Doerr & Lesh's (2003) multilevel principle is the instructional analog of Dienes (1960,1961) dynamic principle which Doerr and Lesh emphasize by saying "teachers most often need to simultaneously address content, pedagogical strategies, and psychological aspects of a teaching and learning situations" (p. 133). This implies that when we give students an open-ended problem, it becomes our responsibility to address as many aspects of the problem (as possible) and ideally mathematize it to fruition by leading into the discovery of structure as demonstrated earlier. In this endeavor the Lakatosian methodology of conjecture-proof-refutation serves as a valuable tool to mathematize unusual problems and bridge the ideal Lakatosian classroom with the actual mathematics classroom!

NOTES

1. Suppose C is a collection of nonempty sets. Then one can *choose* a member from each set in that collection. Therefore, there exists a function f defined on C such that, for each set S in the collection, $f(S)$ is a member of S.

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REFERENCES

Australian Education Council. (1990). *A national statement on mathematics for Australian schools*. Melbourne, VC: Australian Educational Council.

Bohman, T. (1996). A sum packing problem of Erdös and the Conway-Guy sequence. *Proceedings of the American Mathematical Society, 124*(12), 3627-3636.

Brodkey, J.J. (1996). Starting a Euclid club. *Mathematics Teacher, 89*(5), 386-388.

Dienes, Z.P. (1960). *Building up mathematics*. London: Hutchinson Education.

Dienes, Z.P. (1961). *An experimental study of mathematics learning*. New York: Hutchinson & Co Ltd.

Doerr, H. & Lesh, R. (2003). A modeling perspective on teacher development. In R. Lesh & H. Doerr (Eds.), *Beyond Constructivism* (pp.125-140). New Jersey: Lawrence Erlbaum Associates.

English, L.D. (1998). Children's problem posing within formal and informal contexts. *Journal for Research in Mathematics Education, 29*(1), 83-106.

English, L.D. (1999). Assessing for structural understanding in children's combinatorial problem solving. *Focus on Learning Problems in Mathematics, 21*(4), 63-82.

Fawcett, H. P. (1938). *The nature of proof. Thirteenth yearbook of the NCTM*. New York: Bureau of Publications, Teachers College, Columbia University.

Fomin, D., Genkin, S., & Itenberg, I. (1996). *Mathematical Circles (Russian Experience).* American Mathematical Society.

Gardner, M. (1997). *The last recreations*. New York: Springer-Verlag.

Goldbach, C. (1742) Letter to L. Euler, June 7, 1742. Retrieved January 11, 2004 from http://www.mathstat.dal.ca/~joerg/pic/g-letter.jpg

Guy, R.K. (1982). Sets of integers whose subsets have distinct sums, *Theory and practice of combinatorics, Annals of discrete math, 12*, 141-154. North-Holland, Amsterdam.

Hogendijk, J.P. (1996). Een workshop over Iraanse mozaïken. *Nieuwe Wiskrant, 16*(2), 38-42.

Hung, D. (1998) Meanings, contexts and mathematical thinking: the meaning-context model. *Journal of Mathematical Behavior, 16*(3), 311-344.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge, UK: Cambridge University Press.

Lesh, R & Doerr, H. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning and problem solving. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism* (pp. 3-34). New Jersey: Lawrence Erlbaum Associates.

Lewis, B. (2003). Taking perspective. *The Mathematical Gazette, 87*(510), 418-431.

Maher,C.A. & Kiczek, R.D. (2000). Long term building of mathematical ideas related to proof making. *Contributions to Paolo Boero, G.Harel, C. Maher, M.Miyasaki (organisers), Proof and Proving in Mathematics Education*. ICME9 -TSG 12. Tokyo/Makuhari, Japan.

Maher, C. A. & Martino A. M. (1996a) The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27*(2), 194-214.

Maher C. A.& Martino A.M. (1996b) Young children invent methods of proof: The "Gang of Four." In P.Nesher, L.P. Steffe, P. Cobb, B. Greer & J. Goldin (Eds.), *Theories of mathematical learning,* (pp. 431-447). Mahwah, NJ: Lawrence Erlbaum Associates.

Maher, C. A., & Martino, A. M. (1997) Conditions for conceptual change: From pattern recognition to theory posing. In H. Mansfield (Ed.), *Young children and mathematics: concepts and their representations*. Durham, NH: Australian Association of Mathematics Teachers.

Maher ,C.A. & Speiser, B. (1997). How far can you go with block towers? Stephanie's intellectual development. *Journal of Mathematical Behavior 16*(2) (1997), 125-132.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*: Reston, VA: NCTM.

Rotman, B. (1977). *Jean Piaget: Psychologist of the real*. Cornell University Press.

Sriraman, B. (2003a). Mathematical giftedness, problem solving, and the ability to formulate generalizations. *The Journal of Secondary Gifted Education, 14*(3),151-165.

Sriraman, B. (2003b) Can mathematical discovery fill the existential void? The use of conjecture, proof and refutation in a high school classroom. *Mathematics in School*, *32*(2), 2-6.

Sriraman, B. (2004a). Discovering a mathematical principle: The case of Matt. *Mathematics in School, 33*(2), 25-31.

Sriraman, B. (2004b). Reflective abstraction, uniframes and the formulation of generalizations. *Journal of Mathematical Behavior, 23*(2), 205-222.

Sriraman, B. (2004c). Discovering Steiner triple systems through problem solving. *The Mathematics Teacher, 97*(5) 320-326.

Sriraman, B. (2004d). Re-creating the renaissance. In M. Anaya, C. Michelsen (Eds.), *Proceedings of the topic study group 21 – Relations between mathematics and others subjects of art and science*: *The 10th International Congress of Mathematics Education*, Copenhagen, Denmark (pp. 14-19).

Sriraman, B. & Adrian, H. (2004a). The pedagogical value and the interdisciplinary nature of inductive processes in forming generalizations. *Interchange: A Quarterly Review of Education*, *35(*4), 407-422.

Sriraman, B. & English, L. (2004a). Combinatorial mathematics: Research into practice. Connecting research into teaching. *The Mathematics Teacher, 98*(3), 182-191.

Sriraman, B.& Strzelecki, P. (2004a). Playing with powers. *The International Journal for Technology in Mathematics Education*, *11*(1), 29-34.

Van Maanen, J. (1992). Teaching geometry to 11 year old “medieval lawyers.” *The Mathematical Gazette, 76*(475), 37-45.

Wheeler, D. (2001). Mathematisation as a pedagogical tool. *For the Learning of Mathematics, 21*(2), 50-53.