Stat 549

Fall 2010

Properties of Expectation and Variance

Definition of expected value for a discrete random variable:

$$\mu = \mathrm{E}(X) = \sum_{x \in I} x \mathrm{P}(X = x)$$

Properties of expected value:

1. Substitution rule:

$$\mathrm{E}[g(X)] = \sum_{x \in I} g(x) \mathrm{P}(X = x)$$

   Note: in general, it is <u>not</u> true that $\mathrm{E}[g(X)] = g[\mathrm{E}(X)]$. For example, it is <u>not</u> generally true that $\mathrm{E}(X^2) = [\mathrm{E}(X)]^2$.

2. $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$

3. $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$

4. $\mathrm{E}(X_1 + X_2 + \ldots + X_n) = \sum_{i=1}^{n} \mathrm{E}(X_i)$

5. $\mathrm{E}(\sum_{i=1}^{n} a_i X_i + b) = \sum_{i=1}^{n} a_i \mathrm{E}(X_i) + b$

Definition of variance and standard deviation:

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}\left[(X - \mu)^2\right]$$
$$\sigma = SD(X) = \sqrt{\mathrm{Var}(X)}$$

Properties of variance:

6. $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$

   Proof:

$$\begin{aligned} \mathrm{E}\left[(X - \mu)^2\right] &= \mathrm{E}[X^2 - 2\mu X + \mu^2] \\ &= \mathrm{E}(X^2) - 2\mu\mathrm{E}(X) + \mu^2 = \mathrm{E}(X^2) - 2\mu^2 + \mu^2 = \mathrm{E}(X^2) - \mu^2 \end{aligned}$$

7. $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
   Proof:

$$\begin{aligned} \mathrm{Var}(aX + b) &= \mathrm{E}\left([(aX + b) - \mathrm{E}(aX + b)]^2\right) \\ &= \mathrm{E}\left([aX + b - (a\mu + b)]^2\right) \\ &= \mathrm{E}[a^2(X - \mu)^2] = a^2\mathrm{Var}(X) \end{aligned}$$

8. $\text{SD}(aX+b) = |a|\,\text{SD}(X)$

Independence of random variables Two random variables X and Y are said to be independent if $\text{P}(X \in A \text{ and } Y \in B) = \text{P}(X \in A)\text{P}(Y \in B)$ for any two sets of real numbers A and B. That is, the <u>events</u> X in A and Y in B are independent for every choice of the sets A and B. For two discrete random variables this definition can be shown to be equivalent to the following: two discrete random variables X and Y are independent if $\text{P}(X = x \text{ and } Y = y) = \text{P}(X = x)\text{P}(Y = y)$ for all values of x and y.

Some consequences of independence: if X and Y are independent random variables, then

9. $f(X)$ and $g(Y)$ are independent random variables for any functions f and g.
10. $\text{E}(XY) = \text{E}(X)\text{E}(Y)$.
11. $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$
    Proof:

$$\begin{aligned}
\text{Var}(X+Y) &= \text{E}([(X+Y)-(\mu_X+\mu_Y)]^2) = \text{E}([(X-\mu_X)+(Y-\mu_Y)]^2) \\
&= \text{E}[(X-\mu_X)^2] + \text{E}[(Y-\mu_Y)^2] + 2\text{E}[(X-\mu_X)(Y-\mu_Y)] \\
&= \text{Var}(X) + \text{Var}(Y) + 2\text{E}(X-\mu_X)\text{E}(Y-\mu_Y) \text{ by properties 9 and 10} \\
&= \text{Var}(X) + \text{Var}(Y) \text{ because } \text{E}(X-\mu_X) = \text{E}(X) - \mu_X = 0
\end{aligned}$$

The n random variables $X_1, \dots, X_n$ are said to be independent if $\text{P}(X_1 \in A_1, X_2 \in A_2, \dots, X_n \in A_n) = \text{P}(X_1 \in A_1)\text{P}(X_2 \in A_2)\dots\text{P}(X_n \in A_n)$ for any n sets of real numbers $A_1, \dots, A_n$.

If $X_1, \dots, X_n$ are independent random variables, then

12. $\text{Var}(X_1 + X_2 + \ldots + X_n) = \sum_{i=1}^n \text{Var}(X_i)$
13. $\text{Var}\left(\sum_{i=1}^n a_i X_i + b\right) = \sum_{i=1}^n a_i^2 \text{Var}(X_i)$

Special note: It is very important to know how to use result 13 when some of the coefficients are negative. For example, if X and Y are independent, is it true that $\text{Var}(X-Y) = \text{Var}(X) - \text{Var}(Y)$? **NO**. This can't be true because it would imply that $\text{Var}(X-Y)$ could be negative which is impossible. The correct way to calculate $\text{Var}(X-Y)$ is to use result 13:

$$\text{Var}(X-Y) = \text{Var}(X+(-1)Y) = \text{Var}(X) + (-1)^2\text{Var}(Y) = \text{Var}(X) + \text{Var}(Y).$$