General neighbour-distinguishing index of a graph

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Abstract

It is proved that edges of a graph $G$ with no component $K_2$ can be coloured using at most $2 \lceil \log_2 \chi(G) \rceil + 1$ colours so that any two adjacent vertices have distinct sets of colours of their incident edges.

Keywords: Edge colouring; Colour set; General neighbour-distinguishing index

1 Introduction

All graphs we deal with in this paper are simple and finite. Let $G$ be a graph and $k$ a non-negative integer. A (general) $k$-edge-colouring of $G$ is a mapping $\varphi : E(G) \to \bigcup_{i=1}^k \{i\}$. The colour set (with respect to $\varphi$) of a vertex $x \in V(G)$ is the set $S_\varphi(x)$ of colours (values of $\varphi$) of edges incident to $x$. The colouring

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ϕ is neighbour-distinguishing if $S_ϕ(x) \neq S_ϕ(y)$ whenever vertices $x, y$ are adjacent. A neighbour-distinguishing colouring will be frequently shortened to an nd-colouring. The general neighbour-distinguishing index of $G$ is the minimum $k$ in a general $k$-edge-colouring of $G$ that is neighbour-distinguishing, and will be denoted by $\gnd(G)$. If $G$ has a component $K_2$, then $G$ does not have any nd-colouring, hence for the sake of the completeness of the definition in such a case we set $\gnd(G) := \infty$. Having in mind the following evident statement, the analysis of the general neighbour-distinguishing index can be restricted to connected graphs.

**Proposition 1.** If $n \geq 2$ and $G$ is a disconnected graph with components $G_1, \ldots, G_n$, then $\gnd(G) = \max (\gnd(G_i) : i = 1, \ldots, n)$. □

The general neighbour-distinguishing index is a relaxation of two known graph invariants. If $S_ϕ(x) \neq S_ϕ(y)$ is required for any two distinct vertices $x, y$, the corresponding parameter $\chi_0(G)$, called the point-distinguishing chromatic index of $G$, has been introduced by Harary and Plantholt in [3]. The authors proved, among other things, that $\chi_0(K_n) = \lceil \log_2 n \rceil + 1$ for any $n \geq 3$. In spite of the fact that the structure of complete bipartite graphs is simple, it seems that the problem of determining $\chi_0(K_m,n)$ is not easy, especially in the case $m = n$, as documented by papers of Zagaglia Salvi [8], [9], Horňák and Soták [5], [6] and Horňák and Zagaglia Salvi [7].

On the other hand, if only proper nd-colourings are considered, the neighbour-distinguishing index of $G$, in notation $\nd(G)$, is obtained. This invariant has been introduced only recently by Zhang et al. in [10]. It is easy to see that $\nd(C_5) = 5$ and in [10] it is conjectured that $\nd(G) \leq \Delta(G) + 2$ for any connected graph $G \notin \{K_2, C_5\}$. The conjecture has been confirmed by Balister et al. in [1] for bipartite graphs and for graphs $G$ with $\Delta(G) = 3$. Edwards et al. in [2] have shown even that $\nd(G) \leq \Delta(G) + 1$ if $G$ is bipartite, planar, and of maximum degree $\Delta(G) \geq 12$. In the general case a weaker statement $\nd(G) \leq \Delta(G) + 300$ has been proved by Hatami in [4] for all graphs $G$ with $\Delta(G) > 10^{20}$.

For integers $p, q$ we denote by $[p, q]$ the integer interval lower bounded by $p$ and upper bounded by $q$, i.e., $[p, q] := \bigcup_{i=p}^{q} \{i\}$. Let $n$ and $l_1, \ldots, l_n$ be non-negative integers. The concatenation of finite sequences $A_i = (a_{i1}, \ldots, a_{li})$, $i = 1, \ldots, n$, is defined as the sequence $\prod_{i=1}^{n} A_i := (a_{11}, \ldots, a_{1li}, \ldots, a_{n1}, \ldots, a_{nli})$. If $A_i = A$ for each $i \in [1, n]$, we write $A^n$ instead of $\prod_{i=1}^{n} A$. If $n = 0$, $A^n$ is the empty sequence ( ).
Let $G$ be a graph let $x, y \in V(G)$. By $\deg_G(x)$ we denote the degree of $x$ in $G$ and by $d_G(x, y)$ the distance between $x$ and $y$ in $G$. An arm of a tree $T$ is a maximal (non-extendable) subpath $A$ of $T$ such that $\deg_A(x) = \deg_T(x) = 2$ for any internal vertex $x \in V(A)$ and $V(A)$ contains a pendant vertex of $T$. Let $a(T)$ denote the number of arms of $T$. If $T$ is (isomorphic to) an $n$-vertex path $P_n$, then $a(T) = 1$ and $T$ itself is the only arm of $T$. On the other hand, if $\Delta(T) \geq 3$, any arm $A$ of $T$ has one endvertex of degree one, the other of degree at least three and $a(T)$ is equal to the number of pendant vertices of $T$.

The main goal of this paper is to show that $gndi(G) \leq 2\lceil \log_2 \chi(G) \rceil + 1$ for any graph $G$ having no isolated edges.

## 2 Paths, cycles and bipartite graphs

**Proposition 2.** For any graph $G$ the following statements are equivalent:

1. $gndi(G) = 2$.
2. $G$ is bipartite and there is a bipartition $\{X_1 \cup X_2, Y\}$ of $V(G)$ such that $X_1 \cap X_2 = \emptyset$ and any vertex of $Y$ has at least one neighbour in both $X_1$ and $X_2$.

**Proof.** (1) ⇒ (2): Consider an nd-colouring $\varphi : E(G) \to \{1, 2\}$. The only three available non-empty colour sets are $\{1\}, \{2\}$ and $\{1, 2\}$. Since $\{1\} \cap \{2\} = \emptyset$, for any $xy \in E(G)$ exactly one of $S_\varphi(x)$ and $S_\varphi(y)$ is equal to $\{1, 2\}$. Let $Y := \{y \in V(G) : S_\varphi(y) = \{1, 2\}\}$ and let $X_i := \{x \in V(G) : S_\varphi(x) = \{i\}\}$, $i = 1, 2$. Clearly, $X_1 \cap X_2 = \emptyset$, $(X_1 \cup X_2) \cap Y = \emptyset$, any edge of $G$ joins a vertex of $X_1 \cup X_2$ to a vertex of $Y$, and any vertex of $Y$ has at least one neighbour in both $X_1$ and $X_2$.

(2) ⇒ (1): Let the colouring $\varphi : E(G) \to [1, 2]$ be defined so that $\varphi(xy) = i$ if and only if $x \in X_i$ and $y \in Y$, $i = 1, 2$. Then $S_\varphi(x) = \{i\}$ for any $x \in X_i$, $i = 1, 2$, $S_\varphi(y) = \{1, 2\}$ for any $y \in Y$, and so $\varphi$ is neighbour-distinguishing. \hfill \Box

An nd-colouring $\varphi : E(G) \to [1, 3]$ of a bipartite graph $G$ is said to be canonical if there is a canonical ordered bipartition $(X, Y)$ of $V(G)$, one that satisfies $S_\varphi(x) \in S_1 := \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}$ for every $x \in X$ and $S_\varphi(y) \in S_2 := \{\{3\}, \{1, 2\}\}$ for every $y \in Y$. The set $S_1$ has the following important property: whenever $S \in S_1$, then also $S \cup \{3\} \in S_1$. A canonical nd-colouring $\varphi$ of a tree $T$ is 3-canonical if $S_\varphi(v) \neq \{3\}$ for any vertex
$v \in V(T)$ with $\deg_T(v) \geq 2$. A 3-canonical nd-colouring $\varphi$ of a path $P_n$ is $(3, i)$-canonical, $i \in [1, 2]$, if there is a pendant edge $e \in E(P_n)$ such that $\varphi(e) = i$.

**Proposition 3.** Let $n$ be an integer, $n \geq 3$, and let $i \in [1, 2]$.

1. If $n$ is odd, then $\text{gndi}(P_n) = 2$ and there is a $(3, i)$-canonical nd-colouring $\varphi : E(P_n) \rightarrow [1, 2]$.

2. If $n$ is even, then $\text{gndi}(P_n) = 3$ and there is a $(3, i)$-canonical nd-colouring $\varphi : E(P_n) \rightarrow [1, 3]$.

**Proof.** Suppose that $\text{gndi}(P_n) = 2$ and let $\{X_1 \cup X_2, Y\}$ be the bipartition of $V(P_n)$ yielded by Proposition 2. The natural sequence of vertices of $P_n$ (from one endvertex to the other) is an alternating sequence of vertices from $X_1 \cup X_2$ and $Y$ that starts and ends with a vertex of $X_1 \cup X_2$. Therefore $|X_1 \cup X_2| = |Y| + 1$ and $n$ is odd.

1. If $n = 4j - 1$, then the nd-colouring, determined by the sequence $(1, 2)(2, 1, 1, 2)^{j-1}$ of colours of consecutive edges of $P_n$, is both $(3, 1)$-canonical and $(3, 2)$-canonical. If $n = 4j + 1$, the nd-colouring, corresponding to $(1, 2, 2, 1)^j$, is $(3, 1)$-canonical.

2. If $n = 4j$ or $n = 4j + 2$, then the sequence $(3, 2, 1)(1, 2, 2, 1)^{j-1}$ or $(3)(1, 2, 2, 1)^j$, respectively, represents a $(3, 1)$-canonical nd-colouring of $P_n$.

If $\varphi$ is a $(3, 1)$-canonical nd-colouring of $P_n$, then the colouring $\tilde{\varphi}$, defined by $\varphi(e) = 3 \Rightarrow \tilde{\varphi}(e) = 3$ and $\varphi(e) = k \in [1, 2] \Rightarrow \tilde{\varphi}(e) = 3 - k$, is a $(3, 2)$-canonical nd-colouring of $P_n$, and uses the same number of colours as $\varphi$ does. \hfill \Box

**Proposition 4.** Let $n$ be an integer, $n \geq 3$.

1. If $n \equiv 0 \pmod{4}$, then $\text{gndi}(C_n) = 2$.

2. If $n \not\equiv 0 \pmod{4}$, then $\text{gndi}(C_n) = 3$.

**Proof.** Suppose that $\text{gndi}(C_n) = 2$ and let $\{X_1 \cup X_2, Y\}$ be the bipartition of $V(C_n)$ from Proposition 2. Pick a vertex $y \in Y$, take his unique neighbour $x_1 \in X_1$ and consider the natural sequence of vertices of $C_n$ given by the ordered pair $(y, x_1)$ that ends with the other neighbour $x_2 \in X_2$ of $y$. This sequence is built up by concatenating ordered 4-tuples of vertices belonging successively to $Y, X_1, Y$ and $X_2$, hence $n \equiv 0 \pmod{4}$.

The following (cyclic) sequences represent an nd-colouring of $C_n$ with minimum possible number of colours successively for $n = 4j - 1, 4j, 4j + 1$ and $4j + 2$: $(1, 2, 3)(1, 2, 2, 1)^{j-1}$, $(1, 2, 2, 1)^j$, $(1, 2, 2, 3, 1)(1, 2, 2, 1)^{j-1}$, $(1, 2, 3)^2(1, 2, 2, 1)^{j-1}$.

\hfill \Box
Theorem 5. If $T$ is a tree with $|E(T)| \geq 2$, then $\text{gndi}(T) \leq 3$ and there is a 3-canonical nd-colouring of $T$.

Proof. We proceed by induction on $a(T)$. If $a(T) = 1$, there is $n \geq 3$ such that $T \simeq P_n$ and we are done by Proposition 3.

Suppose that $a(T) > 1$ and there is a 3-canonical nd-colouring of an arbitrary tree $T'$ with $a(T') < a(T)$. Consider a pendant vertex $x \in V(T)$ and such a vertex $y \in V(T)$ with $\deg_T(y) \geq 3$ that $d_T(x, y)$ is minimal. The subpath $A$ of $T$ with endvertices $x$ and $y$ is an arm of $T$ and $T' := T - (V(A) - \{y\})$ is a subtree of $T$ with $a(T') = a(T) - 1$ and $|E(T')| \geq 2$.

By the induction hypothesis there is a 3-canonical nd-colouring $\varphi': E(T') \to [1, 3]$. Let $(X', Y')$ be a canonical ordered bipartition of $V(T')$ (there is one corresponding to $\varphi'$). A 3-canonical nd-colouring $\psi: E(T) \to [1, 3]$ will be found as a continuation of $\varphi'$.

(1) $V(A) = \{x, y\}$

(11) If $S_{\varphi'}(y) \neq \{1, 2\}$, then $S_{\varphi'}(y) \in S_1$. Defining $\psi(xy) := 3$ yields $S_{\psi}(y) = S_{\varphi'}(y) \cup \{3\} \in S_1$, $S_{\psi}(x) = \{3\} \in S_2$ and $(X', Y' \cup \{x\})$ is the canonical ordered bipartition of $V(T)$.

(12) If $S_{\varphi'}(y) = \{1, 2\}$, set $\psi(xy) := 3$. Then $S_{\psi}(x) = \{1\} \in S_1$, $S_{\psi}(y) = \{1, 2\} \in S_2$ and $(X' \cup \{x\}, Y')$ is the canonical ordered bipartition of $V(T)$.

(2) Provided that $|V(A)| \geq 3$, let $z$ be the unique neighbour of $y$ in $A$. Since $\deg_T(z) - 1 \geq 2$ and the colouring $\varphi'$ is 3-canonical, there is $i \in S_{\varphi'}(y) \cap [1, 2]$. By Proposition 3 there exists a $(3, i)$-canonical nd-colouring $\varphi : E(A) \to [1, 3]$ with $\varphi(yz) = i$. Clearly, if $(X, Y)$ is the canonical ordered bipartition of $V(A)$, then $y \in X$, $z \in Y$ and $S_{\varphi}(z) = \{1, 2\}$.

(21) If $S_{\varphi'}(y) \neq \{1, 2\}$, let $\psi$ be the common continuation of both $\varphi'$ and $\varphi$. In such a case $S_{\psi}(v) = S_{\varphi'}(v)$ for any $v \in V(T')$, $S_{\psi}(v) = S_{\varphi}(v)$ for any $v \in V(A) - \{y\}$ and the canonical ordered bipartition of $V(T)$ is $(X' \cup X, Y' \cup Y)$.

(22) If $S_{\varphi'}(y) = \{1, 2\}$, then $y \in Y'$.

(221) If $V(A) = \{x, y, z\}$, set $\psi(yz) := 2$ and $\psi(zx) := 3$ to obtain $S_{\psi}(y) = \{1, 2\} \in S_2$, $S_{\psi}(z) = \{2, 3\} \in S_1$ and $S_{\psi}(x) = \{3\} \in S_2$; the canonical ordered bipartition of $V(T)$ is $(X' \cup \{z\}, Y' \cup \{x\})$.

(222) If $|V(A)| \geq 4$, then $A' := A - y$ is a path on $|V(A)| - 1 \geq 3$ vertices. By Proposition 3 there is a $(3, 1)$-canonical nd-colouring $\varphi^- : E(A') \to [1, 3]$ such that $S_{\varphi^-}(z) = \{1\}$; if $(X', Y')$ is the canonical ordered bipartition of $V(A')$, then $z \in X^-$. The continuation $\psi$ of both $\varphi'$ and $\varphi^-$ with $\psi(yz) := 1$ satisfies $S_{\psi}(v) = S_{\varphi'}(v)$ for any $v \in V(T')$, $S_{\psi}(v) = S_{\varphi^-}(v)$ for any $v \in V(A^-)$.
and \((X' \cup X^-, Y' \cup Y^-)\) is the canonical ordered bipartition of \(V(T)\).

**Theorem 6.** If \(G\) is a connected bipartite graph with \(|E(G)| \geq 2\), then \(gndi(G) \leq 3\) and \(G\) has a canonical nd-colouring.

**Proof.** We prove the theorem by induction on the cyclomatic number \(\mu(G) := |E(G)| - |V(G)| + 1\). If \(\mu(G) = 0\), then \(G\) is a tree and we can use Theorem 5. Assume that \(\mu(G) > 0\) and there is a canonical nd-colouring of any connected bipartite graph \(H\) satisfying \(|E(H)| \geq 2\) and \(\mu(H) < \mu(G)\). From \(\mu(G) > 0\) it follows that there is a cycle \(C\) in \(G\) (of an even length). If \(xy \in E(C)\), then by the induction hypothesis for the connected graph \(H := G - xy\) with \(|E(H)| = |E(G)| - 1 \geq 3\) and \(\mu(H) = \mu(G) - 1\) there exists a canonical nd-colouring \(\varphi : E(H) \to [1, 3]\) with a canonical ordered bipartition \((X, Y)\) of \(V(H)\). Without loss of generality we may suppose that \(x \in X\) and \(y \in Y\). Then there is a canonical nd-colouring \(\psi : E(G) \to [1, 3]\) that is a continuation of \(\varphi\) and has the canonical ordered bipartition \((X, Y)\) of \(V(G) = V(H)\).

Namely, if \(S_\varphi(x) \cap S_\varphi(y) \neq \emptyset\), using \(\psi(xy) \in S_\varphi(x) \cap S_\varphi(y)\) leads to \(S_\psi(x) = S_\varphi(x)\) and \(S_\psi(y) = S_\varphi(y)\).

If \(S_\varphi(x) \cap S_\varphi(y) = \emptyset\), there is \(i \in [1, 2]\) such that \(S_\varphi(x) = \{i\}\) and \(S_\varphi(y) = \{3\}\); in such a case setting \(\psi(xy) := 3\) yields \(S_\psi(x) = \{i, 3\} \in S_1\) and \(S_\psi(y) = \{3\} \in S_2\).

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\begin{align*}
3 \text{ Main result}
\end{align*}
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We use the following lemma proved in [1].

**Lemma 7.** If \(G\) is a graph having neither \(K_2\) nor \(K_3\) as a component, then \(G\) can be written as an edge-disjoint union of \(\lceil \log_2 \chi(G) \rceil\) bipartite graphs, each of which has no component \(K_2\).

**Theorem 8.** \(gndi(G) \leq 2\lceil \log_2 \chi(G) \rceil + 1\) for any graph \(G\) without isolated edges.

**Proof.** Because of Proposition 1 we may suppose without loss of generality that \(G\) is connected. If \(G = K_1\), then \(gndi(G) = 0\). For \(G = K_3 = C_3\) Proposition 4 yields \(gndi(G) = 3\). If \(G \notin \{K_1, K_3\}\), put \(r := \lceil \log_2 \chi(G) \rceil\). By Lemma 7 we know that \(G\) can be written as an edge-disjoint union of \(r\) bipartite graphs, each of which has no component \(K_2\). Let \(B_1, \ldots, B_r\) be such an edge-disjoint decomposition of \(G\). By Theorem 6, for any \(i \in [1, r] \)
there is an ordered bipartition \((X_i, Y_i)\) of \(V(B_i)\) and a colouring \(\varphi_i : E(B_i) \rightarrow \{1, 2i, 2i+1\}\) satisfying \(S_{\varphi_i}(x) \subseteq \{2i, \{2i+1\}\}\) for every \(x \in X_i\) and \(S_{\varphi_i}(y) \subseteq \{1\}\) for every \(y \in Y_i\). Then \(\varphi := \bigcup_{i=1}^{r} \varphi_i\), the common continuation of all \(\varphi_i\)'s, is an nd-colouring of \(G\). Indeed, for any edge \(e \in E(G)\) there is \(i \in [1, r]\) such that \(e \in E(B_i)\), and so \(e = xy\) with \(x \in X_i\) and \(y \in Y_i\). Trivially, \(S_{\varphi_i}(x) \subseteq S_\varphi(x)\) and \(S_{\varphi_i}(y) \subseteq S_\varphi(y)\). Since exactly one of the colours \(2i, 2i+1\) is in \(S_\varphi(x)\) and \(S_\varphi(y)\) contains either both colours \(2i, 2i+1\) or none of them, we have \(S_\varphi(x) \neq S_\varphi(y)\). Thus, the colouring \(\varphi : E(G) \rightarrow [1, 2r+1]\) shows that \(gndi(G) \leq 2r + 1 = 2[\log_2 \chi(G)] + 1\).

**Corollary 9.** \(gndi(G) \leq 5\) for any planar graph \(G\) without isolated edges. □

It may be a little bit surprising that \(gndi(I) = 3\) for the icosahedron graph \(I\). In fact, we do not know any planar graph whose general neighbour-distinguishing index is greater than 3.

**Problem 1.** Does there exist a planar graph \(G\) with \(gndi(G) > 3\) ?

**Theorem 10.** \(gndi(K_n) = [\log_2 n] + 1\) for any integer \(n \geq 3\).

*Proof.* In an nd-colouring of \(K_n\) any two distinct vertices must have distinct colour sets. So, using the result of [3] mentioned in the Introduction, \(gndi(K_n) = \chi_0(K_n) = [\log_2 n] + 1\). □

**Corollary 11.** \(gndi(G) \leq 2[\log_2 \Delta(G)] + 1\) for any connected graph \(G \neq K_2\).

*Proof.* If there is \(n \geq 3\) such that \(G \simeq C_n\) or \(G \simeq K_n\), use Proposition 4 or Theorem 10, respectively. Otherwise, by Brooks’ Theorem, \(\chi(G) \leq \Delta(G)\), and the statement follows from Theorem 8. □

As Propositions 3 and 4 show, there are 2-chromatic graphs \(G\) satisfying \(gndi(G) = 2[\log_2 \chi(G)] + 1\). However, we have been unable to find even a graph \(H\) with \(\chi(H) > 2\) and \(gndi(H) > [\log_2 \chi(H)] + 1\).

**Problem 2.** Find a sharp upper bound for \(gndi(G)\) as a function of \(\chi(G)\).

**References**


