Cross-Sperner families

Dániel Gerbner\textsuperscript{a} Nathan Lemons\textsuperscript{b} Cory Palmer\textsuperscript{a}
Balázs Patkós\textsuperscript{a,\dagger} Vajk Szécsi\textsuperscript{b}

\textsuperscript{a}Hungarian Academy of Sciences, Alfréd Rényi Institute of Mathematics, P.O.B. 127, Budapest H-1364, Hungary
\textsuperscript{b}Central European University, Department of Mathematics and its Applications, Nádor u. 9, Budapest H-1051, Hungary

June 14, 2010

Abstract

A pair of families ($\mathcal{F}, \mathcal{G}$) is said to be cross-Sperner if there exists no pair of sets $F \in \mathcal{F}, G \in \mathcal{G}$ with $F \subseteq G$ or $G \subseteq F$. There are two ways to measure the size of the pair ($\mathcal{F}, \mathcal{G}$): with the sum $|\mathcal{F}| + |\mathcal{G}|$ or with the product $|\mathcal{F}| \cdot |\mathcal{G}|$. We show that if $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$, then $|\mathcal{F}| \cdot |\mathcal{G}| \leq 2^{2n-4}$ and $|\mathcal{F}| + |\mathcal{G}|$ is maximal if $\mathcal{F}$ or $\mathcal{G}$ consists of exactly one set of size $\lceil n/2 \rceil$ provided the size of the ground set $n$ is large enough and both $\mathcal{F}$ and $\mathcal{G}$ are non-empty.

1 Introduction

We use standard notation: $[n]$ denotes the set of the first $n$ positive integers, $2^S$ denotes the power set of the set $S$ and $\binom{S}{k}$ denotes the set of all $k$-element subsets of $S$. The complement of a set $F$ is denoted by $\overline{F}$ and for a family $\mathcal{F}$ we write $\overline{\mathcal{F}} = \{ \overline{F} : F \in \mathcal{F} \}$.

One of the first theorems in the area of extremal set families is that of Sperner [15], stating that if we consider a family $\mathcal{F} \subseteq 2^{[n]}$ such that no set $F \in \mathcal{F}$ can contain any other $F' \in \mathcal{F}$, then the number of sets in $\mathcal{F}$ is at most $\binom{n}{\lfloor n/2 \rfloor}$ and equality holds if and only if $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$. Families satisfying the assumption of Sperner’s theorem are called Sperner families or antichains. The celebrated theorem of Erdős,
Ko and Rado [6] asserts that if for a family $\mathcal{G} \subseteq \binom{[n]}{k}$ we have $G \cap G' \neq \emptyset$ for all $G, G' \in \mathcal{G}$ (families with this property are called intersecting), then the size of $\mathcal{G}$ is at most $\binom{n-1}{k-1}$ provided $2k \leq n$.

There have been many generalizations and extensions both to the theorem of Sperner and to the result by Erdős, Ko and Rado (two excellent but not really recent surveys are [4] and [5]). One such generalization is the following: a pair $(\mathcal{F}, \mathcal{G})$ of families is said to be cross-intersecting if for any $F \in \mathcal{F}, G \in \mathcal{G}$ we have $F \cap G \neq \emptyset$. Cross-intersecting pairs of families have been investigated for quite a while and attracted the attention of many researchers [2, 3, 7, 8, 9, 10, 11, 12]. The present paper deals with the analogous generalization of Sperner families that has not been considered in the literature. A pair $(\mathcal{F}, \mathcal{G})$ of families is said to be cross-Sperner if there exists no pair of sets $F \in \mathcal{F}, G \in \mathcal{G}$ with $F \subseteq G$ or $G \subseteq F$. There are two ways to measure the size of the pair $(\mathcal{F}, \mathcal{G})$: either with the sum $|\mathcal{F}| + |\mathcal{G}|$ or with the product $|\mathcal{F}| \cdot |\mathcal{G}|$. We will address both problems.

Clearly, $|\mathcal{F}| + |\mathcal{G}| \leq 2^n$ as by definition $\mathcal{F} \cap \mathcal{G} = \emptyset$. The sum $2^n$ can be obtained by putting $\mathcal{F} = \emptyset, \mathcal{G} = 2^n$. Thus, when considering the problem of maximizing $|\mathcal{F}| + |\mathcal{G}|$ we will assume that both $\mathcal{F}$ and $\mathcal{G}$ are non-empty.

We can reformulate our problem in a rather interesting way. Let $\Gamma_n = (V_n, E_n)$ be the graph with vertex set $V_n = 2^n$ and edge set $E_n = \{(F,G) : F, G \in V_n, F \subseteq G \text{ or } G \subseteq F\}$. Then $\max\{|\mathcal{F}| + |\mathcal{G}|\} = 2^n - c(\Gamma_n)$, where $c(\Gamma_n)$ denotes the vertex connectivity of $\Gamma_n$. Moreover, if we let

$$F(n,m) = \max\{|\mathcal{G}| : \mathcal{G} \subseteq 2^n, \exists \mathcal{F} \subseteq 2^n \text{ with } |\mathcal{F}| = m, (\mathcal{F}, \mathcal{G}) \text{ is cross-Sperner}\}$$

then, denoting by $N_{\Gamma_n}(U)$ the neighborhood of $U$ in $\Gamma_n$, we have

$$F(n,m) = 2^n - m - \min\{|N_{\Gamma_n}(\mathcal{F})| : \mathcal{F} \subseteq V_n, |\mathcal{F}| = m\}.$$ 

Thus determining $F(n,m)$ is equivalent to the isoperimetric problem for the graph $\Gamma_n$.

Let us mention that the cross-Sperner property of the pair $(\mathcal{F}, \mathcal{G})$ is equivalent to $(\mathcal{F}, \overline{\mathcal{G}})$ being cross-intersecting and cross-co-intersecting, i.e. for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ we have $F \cap \overline{G} \neq \emptyset$ and $F \cup \overline{G} \neq [n]$.

The rest of the paper is organized as follows. In Section 2, we consider the problem of maximizing $|\mathcal{F}| + |\mathcal{G}|$ and prove the following theorem.

**Theorem 1.1.** There exists an integer $n_0$ such that if $n \geq n_0$ and the pair $(\mathcal{F}, \mathcal{G})$ is cross-Sperner with $\emptyset \neq \mathcal{F}, \mathcal{G} \subseteq 2^n$, then

$$|\mathcal{F}| + |\mathcal{G}| \leq F(n,1) + 1 = 2^n - 2^{[n/2]} - 2^{[n/2]} + 2,$$

and equality holds if and only if $\mathcal{F}$ or $\mathcal{G}$ consists of exactly one set $S$ of size $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ and the other family consists of all subsets of $[n]$ not contained in $S$ and not containing $S$.
In Section 3, we address the problem of maximizing $|\mathcal{F}| \cdot |\mathcal{G}|$. Our result is the following theorem.

**Theorem 1.2.** If $n \geq 2$ and $(\mathcal{F}, \mathcal{G})$ is cross-Sperner with $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$, then the following inequality holds:

$$|\mathcal{F}| |\mathcal{G}| \leq 2^{2^{n-4}}.$$  

This bound is best possible as shown by $\mathcal{F} = \{F \in 2^{[n]} : \exists F \in \mathcal{F}, n \notin F\}$, $\mathcal{G} = \{G \in 2^{[n]} : n \in G, 1 \notin G\}$.

Finally, Section 4 contains some concluding remarks and open problems.

## 2 Proof of Theorem 1.1

Before we start the proof of Theorem 1.1, let us introduce some notation and state a theorem that we will use in our proof. For a $k$-uniform family $\mathcal{F} \subseteq \binom{[n]}{k}$ let $\Delta \mathcal{F} = \{G \in \binom{[n]}{k-1} : \exists F \in \mathcal{F}, G \subset F\}$ be the shadow of $\mathcal{F}$. The following version of the shadow theorem is due to Lovász [13].

**Theorem 2.1.** [Lovász [13]] Let $\mathcal{F} \subseteq \binom{[n]}{k}$ and let us define the real number $x$ by $|\mathcal{F}| = \left(\frac{x}{k}\right)^k$. Then we have $\Delta \mathcal{F} \geq \left(\frac{x}{k-1}\right)^{k-1}$.

For any $F \in 2^{[n]}$ we have $N_{\Gamma_n}(F) = 2^{|F|} + 2^{n-|F|} - 2$ which is minimized if $|F| = \lceil n/2 \rceil$. This proves $F(n, 1) = 2^n - 2^{\lceil n/2 \rceil} + 1$ as stated in Theorem 1.1.

**Proposition 2.2.** If a pair $(\mathcal{F}, \mathcal{G})$ maximizes $|\mathcal{F}| + |\mathcal{G}|$, then both $\mathcal{F}$ and $\mathcal{G}$ are convex families i.e. $F_1 \subset F \subset F_2$, $F_1, F_2 \in \mathcal{F}$ implies $F \in \mathcal{F}$.

**Proof.** If $F, F_1, F_2$ are as above, then $F$ can be added to $\mathcal{F}$ since any set containing $F$ contains $F_1$ and any subset of $F$ is a subset of $F_2$. \qed

Let $(\mathcal{F}, \mathcal{G})$ be a pair of cross-Sperner families and let $F_0$ and $G_0$ be sets of minimum size in $\mathcal{F}$ and $\mathcal{G}$.

**Proposition 2.3.** If $|F_0| + |G_0| < \lceil n/2 \rceil - 1$, then $|\mathcal{F}| + |\mathcal{G}| < F(n, 1)$.

**Proof.** No set containing $F_0 \cup G_0$ can be a member of $\mathcal{F}$ or $\mathcal{G}$. \qed

As $(\mathcal{F}, \mathcal{G})$ is cross-Sperner if and only if $(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ is cross-Sperner, by taking complements (if necessary) and Proposition 2.3 we may and will assume that $m := |F_0| \geq \lceil n/4 \rceil$. Let us write $\mathcal{F}^* = \{F \in \mathcal{F} : F_0 \subset F\}$. Subsets of $F_0$ are not in $\mathcal{F}$ by the minimality of $F_0$ and by the cross-Sperner property they cannot be in $\mathcal{G}$ either, thus
to prove Theorem 1.1 we need to show that there exist more than $|\mathcal{F}^*|$ many sets that are not contained in $\mathcal{F} \cup \mathcal{G}$ and are not subsets of $F_0$. For any $F^* \in \mathcal{F}^*$ let us define

$$B(F^*) = \{F^* \setminus F_0' : F_0' \subseteq F_0, |F^* \setminus F_0'| < m\}.$$ 

Clearly, for any $F^*_1, F^*_2 \in \mathcal{F}^*$ we have $B(F^*_1) \cap B(F^*_2) = \emptyset$ as they already differ outside $F_0$. By definition, no set in $B := \bigcup_{F^* \in \mathcal{F}^*} B(F^*)$ is a subset of $F_0$ as all sets in $B$ have size smaller than $m$ and $B \cap \mathcal{F} = \emptyset$ by the cross-Sperner property. Thus to prove Theorem 1.1 it is enough to show that $|\mathcal{F}^*| < |B|$.

Note the following three things:

• $|B(F^*)| = \sum_{i=|F^* \setminus F_0|+1}^{m} \binom{m}{i}$,

• $\mathcal{F}^{**} = \{F^* \setminus F_0 : F^* \in \mathcal{F}^*\}$ is downward closed as $\mathcal{F}$ and $\mathcal{F}^*$ are convex,

• $|\mathcal{F}^{**}| = |\mathcal{F}^*|$.

Therefore the following lemma finishes the proof of Theorem 1.1 by choosing $A = \mathcal{F}^{**}$, $k = m$ and $n' = n - |F_0|$.

**Lemma 2.4.** Let $\emptyset \neq A \subseteq 2^{[n]}$ be a downward closed family and $k \geq n'/3$. Then if $n'$ is large enough, the following holds

$$|A| < \sum_{A \in A} \sum_{i=|A|+1}^{k} \binom{k}{i}. \tag{1}$$

**Proof.** Let $a_i = |\{A \in A : |A| = i\}|$ and $w(j) = \sum_{i=j+1}^{k} \binom{k}{i}$. Then we can formulate (1) in the following way:

$$\sum_{j=0}^{n'} a_j < \sum_{j=0}^{n'} a_j w(j). \tag{2}$$

Let $x$ be defined by $a_{k-1} = \binom{x}{k-1}$. By Theorem 2.1 if $j \leq k - 1$ then $a_j \geq \binom{x}{j}$. If we replace $a_j$ by $\binom{x}{j}$ in (2), then the LHS decreases by $a_j - \binom{x}{j}$ and the RHS decreases by $(a_j - \binom{x}{j})w(j)$, which is larger. If $j \geq k - 1$, then $a_j \leq \binom{x}{j}$ again by Theorem 2.1. If we replace $a_j$ by $\binom{x}{j}$ in (2), then the LHS increases while the RHS does not change (as for $j \geq k$ we have $w(j) = 0$). Hence it is enough to prove

$$\sum_{j=0}^{n'} \binom{x}{j} < \sum_{j=0}^{n'} \binom{x}{j} w(j). \tag{3}$$

4
First we prove (3) for \( x = n' \). In this case the LHS is \( 2^{n'} \) while the RHS is monotone increasing in \( k \), thus it is enough to prove for \( k = \lceil n/3 \rceil \). We will estimate the RHS from below by considering only one term of the sum. Clearly, \((n')_j w(j) \geq (n')_{j+1} (n'/3)\). Let us write \( j = \alpha n' \) for some \( 0 \leq \alpha \leq 1/3 \). Then by Stirling’s formula we obtain

\[
\binom{n'}{j} \left( \frac{n'}{n'} \right) = \binom{n'}{\alpha n'} \left( \frac{n' \alpha}{n' + 1} \right) = \Theta \left( \frac{1}{n' \left( \alpha^{2\alpha(1-\alpha)(1/3)} \right)^{1/3}} \right) ^{n'}.
\]

The value of the fraction in parenthesis is larger than 2 for, say, \( \alpha = 2/9 \), thus (3) holds if \( n' \) is large enough and \( x = n' \).

To prove (3) for arbitrary \( x \), let \( c = (\binom{x}{k-1})/(\binom{n'}{k-1}) \). If \( j > k - 1 \), then \( c > (\binom{x}{j})/(\binom{n'}{j}) \), while if \( j < k - 1 \), then \( c < (\binom{x}{j})/(\binom{n'}{j}) \). By the \( x = n' \) case we know

\[
\sum_{j=0}^{n'} c(\binom{n'}{j}) < \sum_{j=0}^{n'} c(\binom{n'}{j}) w(j).
\]

Let us replace \( c(\binom{n'}{j}) \) by \( (\binom{x}{j}) \) in this inequality. If \( j > k - 1 \), then the LHS decreases and the RHS does not change. If \( j = k - 1 \) none of the sides change by definition of \( c \). If \( j < k - 1 \), both sides increase, and the RHS increases more as \( w(j) \geq 1 \) for all \( 0 \leq j \leq k-1 \). Hence the inequality holds and gives back (3), which finishes the proof of the lemma.

We believe that Theorem 1.1 is valid for all \( n \), but unfortunately Lemma 2.4 fails for small values of \( n \).

### 3 Proof of Theorem 1.2

In this section we prove Theorem 1.2. Our main tool will be the following special case of the Four Functions Theorem of Ahlswede and Daykin [1]. To state their result for any pair \( \mathcal{A}, \mathcal{B} \) of families let us write \( \mathcal{A} \cap \mathcal{B} = \{ A \cap B : A \in \mathcal{A}, B \in \mathcal{B} \} \) and \( \mathcal{A} \cup \mathcal{B} = \{ A \cup B : A \in \mathcal{A}, B \in \mathcal{B} \} \).

**Theorem 3.1.** [Ahlswede-Daykin, [1]] For any pair \( \mathcal{A}, \mathcal{B} \) of families we have

\[
|\mathcal{A}| |\mathcal{B}| \leq |\mathcal{A} \cap \mathcal{B}| |\mathcal{A} \cup \mathcal{B}|.
\]

To prove Theorem 1.2 we will need the following lemma.

**Lemma 3.2.** If \( (\mathcal{F}, \mathcal{G}) \) is a pair of cross-Sperner families, then the families \( \mathcal{F}, \mathcal{G} \), \( \mathcal{F} \cap \mathcal{G} \) and \( \mathcal{F} \cup \mathcal{G} \) are pairwise disjoint.
Proof. \( F \) and \( G \) are disjoint as some set \( F \in F \cap G \) is a subset of itself and thus contradicts the cross-Sperner property. \( F \) and \( G \) are both disjoint from \( F \land G \) and \( F \lor G \) as \( F \cap G \subseteq F, G \) and \( F, G \subseteq F \cup G \). Finally, \( F \land G \) and \( F \lor G \) are disjoint as \( F_1 \cap G_1 = F_2 \cup G_2 \) would imply \( F_1 \subseteq G_2 \).

Now we are able to prove Theorem 1.2.

Proof. Let \((F, G)\) be a cross-Sperner pair of families. Clearly, if \(|F| + |G| \leq 2^{n-1}\), then the statement of the theorem holds. But if \(|F| + |G| > 2^{n-1}\), then by Lemma 3.2 we have \(|F \land G| + |F \lor G| < 2^{n-1}\) and thus by Theorem 3.1 we obtain \(|F||G| \leq |F \land G||F \lor G| \leq 2^{2n-4}\).

Corollary 3.3. For \(n \geq 2\), we have \(F(n, 2^{n-2}) = 2^{n-2}\).

4 Concluding remarks and open problems

One might wonder whether it changes the situation if we allow sets to belong to both \( F \) and \( G \) and we modify the definition of cross-Sperner families so that only pairs \( F \in F, G \in G \) with \( F \not\subseteq G \) or \( G \not\subseteq F \) are forbidden. It is easy to see that the situation is the same when considering \(|F| + |G|\). To prove that \(|F| + |G| \leq 2^n\) let us write \( C = F \cap G \) and if it is not empty, then \( D(C) := \{ C \setminus C' : C, C' \in C \} \) is disjoint both from \( F \) and \( G \) and a result by Marica and Schönheim [14] tells us that \(|D(C)| \geq |C|\). Note that the proof of Theorem 1.1 works in this case as well giving the upper bound \(|F| + |G| \leq F(n, 1) + 2\).

Although \( F(n, m) \) is not known for most values, it is natural to generalize the problem to \( k \)-tuples of families: \( F_1, F_2, ..., F_k \) is said to be cross-Sperner if for any \( 1 \leq i < j \leq k \) there is no pair \( F \in F_i \) and \( F' \in F_j \) with \( F \subseteq F' \) or \( F' \subseteq F \). One can consider the problems of maximizing \( \sum_{i=1}^{k} |F_i| \) and \( \prod_{i=1}^{k} |F_i| \). In the former case we need the extra assumption that all \( F_i \) are non-empty as otherwise the trivial upper bound \( 2^n \) is tight.

When maximizing the sum, it is natural to conjecture that in the best possible construction all but one family consists of one single set. By the cross-Sperner property, these sets together must form a Sperner family, therefore it might turn out to be useful to introduce

\[ F^*(n, m) = \max\{|G| : G \subseteq 2^{[n]}, \exists F \subseteq 2^{[n]} \text{ with } |F| = m, (F, G) \text{ is cross-Sperner, } F \text{ is Sperner} \}. \]
Problem 4.1. Under what conditions is it true that if $F_1, F_2, \ldots, F_k$ form a $k$-tuple of non-empty cross-Sperner families, then
\[ \sum_{i=1}^{k} |F_i| \leq k - 1 + F^*(n, k - 1)? \]

Concerning maximizing the product of the $|F_i|$, by Theorem 1.2 one obtains that
\[ \prod_{i=1}^{k} |F_i| = \left( \prod_{1 \leq i < j \leq k} |F_i||F_j| \right)^{\frac{1}{k-1}} \leq 2^{kn-2k}. \]

We conjecture that the following construction is optimal: let $l = l(k)$ be the smallest positive integer so that $k \leq \left( \frac{l}{l/2} \right)$. Then there exists a Sperner family $S = \{S_1, \ldots, S_k\} \subset 2^{|l|}$ of size $k$. Put $F_i = \{F \subseteq [n] : F \cap [l] = S_i\}$. Clearly, the $F_i$ form a $k$-tuple of cross-Sperner families and we have $\prod_{i=1}^{k} |F_i| = 2^{k(n-l)}$. Unfortunately, already for $l = 3$ there is a gap of a factor of 8 between the upper bound and the size of our construction.

Conjecture 4.2. If $F_1, F_2, \ldots, F_k \subseteq 2^{[n]}$ form a $k$-tuple of cross-Sperner families, then
\[ \prod_{i=1}^{k} |F_i| \leq 2^{k(n-l)}, \]
where $l$ is the least positive integer with $\left( \frac{l}{l/2} \right) \geq k$.

References


