INSTRUCTIONS: There is no required minimum number of problems to be solved but please do as many of the following problems as time permits. Quality work is more important than quantity and it is not expected that all problems be attempted; five solutions might be a reasonable goal. Succinct solutions displaying insight are most desirable. Partial solutions and/or commentary are also encouraged. Calculators are permitted.

1. What is the one’s digit of $1999^{1998^{1997^{\cdots^{2}}}}$?

2. A quart of low-fat milk is 2% butterfat. This is advertised as 38% less fat than whole milk. What should be the advertised reduction for 1% butterfat?

3. If four Americans, 3 Frenchmen, and 3 Englishmen are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other?

4. Assume that medical science has a test for cancer which is 90% accurate for both those who do and those who do not have cancer. Also assume that 5% of the population tested actually has cancer. If an individual takes this cancer test and the result is positive (i.e., the test says the person has cancer) what is the probability the person really does have cancer?

5. **An SAT problem**: There are three roads connecting Littleton to Quincy, two connecting Quincy to Oroville and one connecting Oroville to Littleton. (No one way roads.) Martina wants to travel from Littleton to Oroville and return going through Quincy only once and travelling each segment of road at most once.
   - In how many ways can she make the round trip?
   - If the destination is Quincy, how many ways are there?
6. Use the graph of the function $f$ given below to estimate the following as accurately as you can.

$$
\begin{array}{ccc}
\text{a. } f'(6) & \text{b. } f'(3) & \text{c. } \int_0^9 f(x) \, dx \\
\end{array}
$$

7. Find three distinct natural numbers such that the sum of their reciprocals is 1. Find all such triples. What if one drops the requirement that the numbers be distinct?

8. Can you determine all real $2 \times 2$ matrices $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$S^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

9. Suppose $m$ and $n$ are odd integers. Prove that the equation $x^2 + 2mx + 2n = 0$ has no rational roots.

10. Let $f$ be a continuous function on the reals, and define $F$ on the positive reals by

$$F(x) = \int_0^x f(t) \, dt.$$ 

Find the derivative $F'$ of $F$.

11. Find the exact value of

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}.$$ 

Show your work. Give some reasoning beyond the fact that in successive approximate calculations, the values appear to approach your answer.

12. A map $f : \mathbb{Q} \to \mathbb{Q}$ (of the rational numbers) is called a ring homomorphism if it satisfies

$$f(1) = 1, \quad f(a + b) = f(a) + f(b), \quad f(a \cdot b) = f(a) \cdot f(b)$$

for all $a, b \in \mathbb{Q}$.

Show that $\mathbb{Q}$ admits exactly one ring homomorphism, namely the identity map.