1991 LENNES EXAMINATION
Department of Mathematical Sciences
THE UNIVERSITY OF MONTANA
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INSTRUCTIONS: There are 11 problems. You are NOT expected to do all of them. Do as many as possible in the time allowed. However, the quality of your work and the insight you display are more important than mere numbers of problems done. Do not hesitate to give partial solutions if you cannot find a complete solution. GOOD LUCK!

1. A dealer bought an article for $7, sold it for $8, bought it back for $9 and finally sold it for $10. How much profit did she make?

2. Let $x$ and $y$ be positive integers such that $3x + 7y$ is divisible by 11. Show that $4x - 9y$ must also be divisible by 11.

3. Prove that a parcel sufficiently large can be mailed by using only $7\ell$ and $13\ell$ stamps. How large is large enough? Is there a generalization?

4. If $m$ and $n$ are odd integers, prove that the equation
   \[ x^2 + 2mx + 2n = 0 \]
   has no rational roots.

5. Let $i$ denote the complex number satisfying $i^2 = -1$. Given that $-i$ is a solution to the equation
   \[ p(z) = iz^3 - 2z^2 - 1 = 0 \]
   find all the other solutions to $p(z) = 0$.

6. Show that, in a group of six people, there is a subset of three people who are mutual acquaintances, or there is a subset of three people who are total strangers.

7. Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$ for $n \geq 1$. Show that $\{a_n\}$ is an increasing sequence which is bounded above. Determine the limit of $a_n$ as $n \to \infty$. 

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8. Define functions \( f \) and \( g \) on \([0, 2]\) by
\[
 f(x) = \max\{x, x^2\} \\
g(x) = \min\{x, x^2\}.
\]
Calculate \( \int_0^2 g(x) \cdot e^{f(x)} \, dx \).

9. Define a function \( F : \mathbb{R} \rightarrow \mathbb{R} \) by 
\[
 F(x) = \int_0^x \sin(\sqrt{t}) \, dt.
\]
a) Find the derivative, \( F' \), of \( F \).
b) Find as simple an expression as possible for \( F(x) \).
c) Calculate \( F(\pi) \).

10. A function \( u : \mathbb{R}^2 \rightarrow \mathbb{R} \) is defined by
\[
 u(x, y) = \min\{x^2 y, y\}, \quad \forall (x, y) \in \mathbb{R}^2.
\]
Sketch the level curve \( \{(x, y) \in \mathbb{R}^2 \mid u(x, y) = 1\}\) in the \( xy\)-plane.

11. Three fishermen go fishing in the middle of a stormy night. They catch some fish, then land on a desert island and go to sleep. Later, one of them wakes up and he says to himself, “I’ll take my one-third of the fish and I’ll go away.” So he divides all the fish into three equal parts. He finds one fish extra. He throws the extra fish into the sea, takes his one-third, and goes. Next, the second fisherman gets up. He doesn’t know the first one has departed. He too divides the remaining catch into three equal parts. He finds one fish extra, throws it into the sea and leaves with his one-third share. Finally, the third fisherman gets up. He does not know what the others have done, but also decides to take his third and leave early. He, in his turn, finds one fish extra which he throws into the sea.

**What Is The Minimum Number Of Fish?**