1989 Lennes Examination
Department of Mathematical Sciences
The University of Montana

Directions:

• Attempt as many problems as you wish.
• Incomplete solutions will receive partial credit.
• Care in presenting results is particularly desirable.

1. A Centennial/Bicentennial problem: Subject to the conditions $a \geq 1789$, $b \leq 1889$ and $c \leq 1999$, find the maximum value of the sum of the roots of the equation $a x^2 - b x + c = 0$.
   For which values of $a, b, c$ does the maximum occur?

2. Let $R$ be the region consisting of points $(x, y)$ of the Cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$.
   Sketch the region $R$ and find its area.

3. a) Find a constant $k$ and a function $f$ so the area of a circle of radius $r$ is expressed by the integral $k \int_0^r f(x) \, dx$.
   b) Find a constant $k$ and a function $f$ so the volume of a sphere of radius $r$ is expressed by the integral $k \int_0^r f(x) \, dx$.
   c) Find a constant $k$ and a function $f$ so the “volume” of the hypersphere $x^2 + y^2 + z^2 + w^2 \leq r^2$ in four dimensional Euclidean space is expressed by the integral $k \int_0^r f(x) \, dx$. 
4. We form a sequence \( P_1, P_2, P_3, P_4, \ldots \) of pentagons of dots as follows:

\[ P_1 \quad P_2 \quad P_3 \]

(The pattern is that of going from \( P_2 \) to \( P_3 \): Extend two adjacent sides and then make the other three sides have the same configuration as those two.)

Let \( a_n \) be the number of dots in \( P_n \) (so \( a_1 = 1 \), \( a_2 = 5 \), \( a_3 = 12 \)). Find a closed expression for \( a_n \).

5. For which values of \( p > 0 \) does \( \lim_{n \to \infty} \left( n! \right)^{-n^p} \) exist? (Since \( n! = 1 \cdot 2 \cdot \ldots \cdot n \), \( \log(n!) = \log(1) + \log(2) + \ldots + \log(n) \).

6. Given four numbers, one can get six (not necessarily all different) numbers by adding the original numbers in pairs. For instance, if the original numbers were \(-1, 3, 5, 9\), then the resulting list of six numbers is \(2, 4, 8, 8, 12, 14\). We shall say that a list of six numbers obtained this way is “compatible”.

   a) Show that \(4, 17, 25, 26, 36, 48\) is not compatible.

   b) Show that \(4, 17, 25, 27, 35, 48\) is compatible and find all possible lists of four numbers which give rise to it.

7. Every inhabitant of the island of Logica is either a knight who always tells the truth or a knave who always lies. I meet four Logicans named A, B, C, D and three of them make the following assertions:

   A: “B is a knave.”
   B: “The number of knaves among us is odd.”
   C: “D is a xark.”

I know that “xark” means either “knave” or “knight” in the Logican language, but I cannot remember which.

Is A a xark? Prove your answer.