Mathematical Statistics **II**
Spring 2009
Solutions to the in-class assignments (03/25/09).

1. *Lemma*: If $f_X(x|\theta)$ is twice differentiable with respect to θ and it satisfies

$$\frac{d}{d\theta}E_\theta\left(\frac{\partial}{\partial\theta}\ln f_X(x|\theta)\right) = \int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}\left[\left\{\frac{\partial}{\partial\theta}\ln f_X(x|\theta)\right\}f_X(x|\theta)\right]\,dx$$

then

$$E\left[\left(\frac{\partial}{\partial\theta}\ln f_X(X|\theta)\right)^2\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f_X(X|\theta)\right].$$

Proof. In the continuous case, but the discrete case can be handled in a similar manner, we know that

$$\int_{-\infty}^{\infty} f_X(x|\theta)\,dx = 1$$

and, by taking the derivative with respect to θ (assuming we can switch the order of derivative with respect to θ and the integral with respect to x derivative under the integral sign),

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}f_X(x|\theta)\,dx = 0.$$

The latter expression can be rewritten as

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}f_X(x|\theta)\,\frac{f_X(x|\theta)}{f_X(x|\theta)}\,dx = \int_{-\infty}^{\infty}\frac{\frac{\partial}{\partial\theta}f_X(x|\theta)}{f_X(x|\theta)}\,f_X(x|\theta)\,dx = 0$$

or, equivalently,

$$\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta}\ln f_X(x|\theta)\right]f_X(x|\theta)\,dx = 0.$$

If we differentiate again (switching the order of differentiation and integration), it follows that

$$\int_{-\infty}^{\infty}\left[\left(\frac{\partial^2}{\partial\theta^2}\ln f_X(x|\theta)\right)f_X(x|\theta) + \left(\frac{\partial}{\partial\theta}\ln f_X(x|\theta)\right)\frac{\partial}{\partial\theta}f_X(x|\theta)\right]dx = 0.$$

Therefore,

$$\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta}\ln f_X(x|\theta)\right)\frac{\partial}{\partial\theta}f_X(x|\theta)dx = -\int_{-\infty}^{\infty}\left(\frac{\partial^2}{\partial\theta^2}\ln f_X(x|\theta)\right)f_X(x|\theta)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta}\ln f_X(x|\theta)\right)\frac{\frac{\partial}{\partial\theta}f_X(x|\theta)}{f_X(x|\theta)}\,f_X(x|\theta)\,dx = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f_X(X|\theta)\right]$$

$$\Rightarrow \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta}\ln f_X(x|\theta)\right]^2 f_X(x|\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f_X(X|\theta)\right]$$

$$\Rightarrow E\left[\left(\frac{\partial}{\partial\theta}\ln f_X(X|\theta)\right)^2\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f_X(X|\theta)\right]$$

as was to be shown. □

Theorem: Let $\hat{\theta}$ be the maximum likelihood estimator of θ obtained by solving

$$\sum_{i=1}^{n} \frac{\partial}{\partial \theta} \ln f_X(x_i|\theta) = 0.$$

Then the distribution of $n^{1/2}[I(\theta)]^{1/2}(\hat{\theta} - \theta)$ will be approximately a standard normal distribution.

2. Suppose that $X_1, X_2, \ldots, X_n$ form a random sample from a normal distribution for which the mean is 0, and the standard deviation σ is unknown ($\sigma > 0$). Show that

$$\hat{\sigma} = \left[\frac{1}{n}\sum_{i=1}^{n} {X_i}^2\right]^{1/2}$$

is an *asymptotically efficient estimator* and that $\hat{\sigma} \sim N(\sigma, \frac{\sigma^2}{n})$.

Proof. Let us first find the MLE of σ. Setting the log-likelihood function equal to zero:

$$\begin{aligned}
\ell(\sigma|x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{\partial}{\partial \sigma} \ln f_X(x_i|\sigma) &= 0 \\
\sum_{i=1}^{n} \frac{\partial}{\partial \sigma} \ln \frac{1}{\sqrt{2\pi}\ \sigma} \exp\left\{-\frac{1}{2}\left(\frac{x_i}{\sigma}\right)^2\right\} &= 0 \\
\sum_{i=1}^{n} \frac{\partial}{\partial \sigma} \left[-\frac{1}{2}\ln 2\pi - \ln \sigma - \frac{1}{2}\left(\frac{x_i}{\sigma}\right)^2\right] &= 0 \\
\sum_{i=1}^{n} \left(-\frac{1}{\sigma} + \frac{{x_i}^2}{\sigma^3}\right) &= 0 \\
\sum_{i=1}^{n} \frac{{x_i}^2}{\sigma^3} &= \frac{n}{\sigma} \\
\frac{1}{n}\sum_{i=1}^{n} {x_i}^2 &= \sigma^2
\end{aligned}$$

and so

$$\hat{\sigma} = \left[\frac{1}{n}\sum_{i=1}^{n} {X_i}^2\right]^{1/2}.$$

We can readily verify that this is a global maximum point!

The Fisher information in a single observation is

$$\begin{aligned}
I(\sigma) &= -E\left[\frac{\partial^2}{\partial \sigma^2} \ln f_X(X|\sigma)\right] \\
&= -E\left[\frac{n}{\sigma^2} - 3\frac{X^2}{\sigma^4}\right] \\
&= -\frac{n}{\sigma^2} + \frac{3}{\sigma^4}\ E(X^2) \\
&= -\frac{n}{\sigma^2} + \frac{3}{\sigma^4}\ \sigma^2 = \frac{2}{\sigma^2}.
\end{aligned}$$

By the asymptotic theory of MLE, it follows that

$$n^{1/2}I(\sigma)^{1/2}(\hat{\sigma}-\sigma) = n^{1/2}\left[\frac{2}{\sigma^2}\right]^{1/2}(\hat{\sigma}-\sigma) \overset{\text{Approx}}{\sim} N(0,1).$$

Therefore $\hat{\sigma}$ has an asymptotic normal distribution with mean σ and variance $\frac{\sigma^2}{2n}$ as was to be shown. Since the MLE attains the CRLB as $n \to \infty$, it is said that $\hat{\sigma}$ is an asymptotically efficient estimator.

$\square$