Mathematical Statistics **II**
Spring 2009
Solutions to the in-class assignments (03/20/09).

1. Suppose that $X_1, \ldots, X_n$ form a random sample from a Poisson distribution for which the mean θ is unknown ($\theta > 0$). We know that

$$\mu = E(X) = \theta, \quad \sigma^2 = \mathrm{Var}(X) = \theta, \quad \overline{X}_n \xrightarrow{p} \theta \quad \text{and} \quad S_n^2 \xrightarrow{p} \theta.$$

Also we know that

$$E(\overline{X}_n) = \theta \qquad \text{and} \qquad E(S_n^2) = \sigma^2 = \theta.$$

Thus, it follows that $\overline{X}_n$ and S_n^2 are both unbiased and consistent estimators of θ. Show that $\overline{X}_n$ is uniformly better than S_n^2. That is, show that $\mathrm{Var}(\overline{X}_n) \le \mathrm{Var}(S_n^2)$.

Solution. Let $Y = \sum_{i=1}^n X_i$, then $Y \sim \text{Poisson}(n\theta)$ and $\frac{1}{n}Y = \overline{X}_n$. Thus

$$\mathrm{Var}(\overline{X}_n) = \mathrm{Var}\left(\tfrac{1}{n}Y\right) = \frac{n\theta}{n^2} = \frac{\theta}{n}.$$

When $E(X^4) < \infty$, the variance of S_n^2 can be expressed as

$$\mathrm{Var}(S_n^2) = \frac{1}{n}\left(\mu_4 - \left(\frac{n-3}{n-1}\right)\sigma^4\right)$$

where $\mu_4 = E(X-\mu)^4$. So in this case, where $X \sim \text{Poisson}(\theta)$,

$$\begin{aligned}\mu_4 = E(X-\theta)^4 &= E\left[X^4 - 4\theta X^3 + 6\theta^2 X^2 - 4\theta^3 X + \theta^4\right] \\ &= E(X^4) - 4\theta E(X^3) + 6\theta^2 E(X^2) - 4\theta^3 E(X) + \theta^4.\end{aligned}$$

Let $M_X(t)$ denote the moment generating function of X, then

$$M_X(t) = e^{\theta(e^t - 1)}.$$

Then

$$E(X) = M_X'(t)|_{t=0} = \theta e^{(e^t-1)\theta + t}\Big|_{t=0} = \theta,$$

$$E(X^2) = M_X''(t)|_{t=0} = \theta\left(e^t\theta + 1\right)e^{(e^t-1)\theta+t}\Big|_{t=0} = \theta(\theta+1) = \theta^2 + \theta,$$

$$E(X^3) = M_X'''(t)|_{t=0} = \theta^2 e^{(e^t-1)\theta+2t} + \theta\left(e^t\theta+1\right)^2 e^{(e^t-1)\theta+t}\Big|_{t=0} = \theta^2 + \theta(\theta+1)^2 = \theta^3 + 3\theta^2 + \theta,$$

and

$$\begin{aligned}E(X^4) &= M_X^{(4)}(t)|_{t=0} \\ &= \theta^2\left(e^t\theta+2\right)e^{(e^t-1)\theta+2t} + 2\theta^2\left(e^t\theta+1\right)e^{(e^t-1)\theta+2t} + \theta\left(e^t\theta+1\right)^3 e^{(e^t-1)\theta+t}\Big|_{t=0} \\ &= \theta^2(\theta+2) + 2\theta^2(\theta+1) + \theta(\theta+1)^3 \\ &= \theta^4 + 6\theta^3 + 7\theta^2 + \theta.\end{aligned}$$

Substituting, we have that

$$\begin{aligned}
\operatorname{Var}(S_n^2) &= \frac{1}{n}\left(\mu_4 - \left(\frac{n-3}{n-1}\right)\sigma^4\right) \\
&= \frac{1}{n}\Bigg((\theta^4 + 6\theta^3 + 7\theta^2 + \theta) - 4\theta(\theta^3 + 3\theta^2 + \theta) + 6\theta^2(\theta^2 + \theta) - 4\theta^3(\theta) + \theta^4 - \left(\frac{n-3}{n-1}\right)\theta^2\Bigg) \\
&= \frac{1}{n}\Bigg(\theta^4 + 6\theta^3 + 7\theta^2 + \theta - 4\theta^4 - 12\theta^3 - 4\theta^2 + 6\theta^4 + 6\theta^3 - 4\theta^4 + \theta^4 - \left(\frac{n-3}{n-1}\right)\theta^2\Bigg) \\
&= \frac{1}{n}\left(3\theta^2 - \frac{(n-3)\theta^2}{n-1} + \theta\right) = \frac{1}{n}\left(\frac{(3n-3-n+3)\theta^2}{n-1} + \theta\right) \\
&= \frac{2\theta^2}{n-1} + \frac{\theta}{n}.
\end{aligned}$$

As

$$\operatorname{Var}(S_n^2) = \frac{2\theta^2}{n-1} + \operatorname{Var}(\overline{X}_n),$$

$\operatorname{Var}(S_n^2) \geq \operatorname{Var}(\overline{X}_n)$ for all $\theta > 0$. □