Mathematical Statistics **II**
Spring 2009
Solutions to the in-class assignments (03/16/09).

1. Suppose that the random variable X has a continuous distribution. Show that $E|X-b|$ is minimum when b is chosen to be the median of X.

Solution. Recall that, for each random variable X, a median of the distribution of X is defined as a number m such that $Pr(X \leq m) \geq 1/2$ and $Pr(X \geq m) \geq 1/2$. From the definition of absolute value

$$|x-b| = \begin{cases} -(x-b) & \text{for } x < b, \\ (x-b) & \text{for } x \geq b \end{cases}.$$

Hence,

$$\begin{aligned} E(|X-b|) &= \int_{-\infty}^{\infty} |x-b| f_X(x)\ dx \\ &= \int_{-\infty}^{b} -(x-b) f_X(x)\ dx + \int_{b}^{\infty} (x-b) f_X(x)\ dx \\ &= \int_{-\infty}^{b} -x f_X(x)\ dx + b\int_{-\infty}^{b} f_X(x)\ dx + \int_{b}^{\infty} x f_X(x)\ dx - b\int_{b}^{\infty} f_X(x)\ dx. \end{aligned}$$

Define the function $Q(b) = E(|X-b|)$ as

$$Q(b) = b\int_{-\infty}^{b} f_X(x)\ dx - \int_{-\infty}^{b} x f_X(x)\ dx - \int_{\infty}^{b} x f_X(x)\ dx + b\int_{\infty}^{b} f_X(x)\ dx,$$

then $Q(b)$ is the function to be minimized. Taking the derivative:

$$\begin{aligned} Q'(b) &= \frac{d}{db} b\int_{-\infty}^{b} f_X(x)\ dx - \frac{d}{db}\int_{-\infty}^{b} x f_X(x)\ dx - \frac{d}{db}\int_{\infty}^{b} x f_X(x)\ dx + \frac{d}{db} b\int_{\infty}^{b} f_X(x)\ dx \\ &= \int_{-\infty}^{b} f_X(x)\ dx + b f_X(b) - b f_X(b) - b f_X(b) + \int_{\infty}^{b} f_X(x)\ dx + b f_X(b) \\ &= \int_{-\infty}^{b} f_X(x)\ dx + \int_{\infty}^{b} f_X(x)\ dx. \end{aligned}$$

To locate the extrema, set $Q'(b)$ (the derivative) equal to zero and solve,

$$Q'(b) = \int_{-\infty}^{b} f_X(x)\ dx + \int_{\infty}^{b} f_X(x)\ dx = 0 \quad \Longrightarrow \quad \int_{-\infty}^{b} f_X(x)\ dx = \int_{b}^{\infty} f_X(x)\ dx.$$

As

$$\int_{-\infty}^{b} f_X(x)\ dx + \int_{b}^{\infty} f_X(x)\ dx = \int_{-\infty}^{\infty} f_X(x)\ dx = 1$$

it follows that

$$\int_{-\infty}^{b} f_X(x)\ dx = \int_{b}^{\infty} f_X(x)\ dx = \frac{1}{2}.$$

Therefore the function takes an extreme value when b is the median of X. To show that this value is minimum, apply the second derivative test. The second derivative of Q is:

$$\begin{aligned} Q''(b) &= \frac{d}{db}\int_{-\infty}^{b} f_X(x)\ dx + \frac{d}{db}\int_{\infty}^{b} f_X(x)\ dx \\ &= f_X(b) + f_X(b). \end{aligned}$$

The second derivative is positive as $f_X(x)$ is a non-negative function (certainly positive at the median). Hence, the function $Q(b)$ is convex when b is the median and therefore $E(|X - b|)$ is minimum when is b chosen to be the median of X. □