

Mathematical Statistics **II**

Spring 2009

Solutions to the in-class assignments (03/13/09).

1. Suppose that X_1, X_2, \dots, X_n form a random sample from a Poisson distribution for which the value of the mean λ is unknown ($\lambda > 0$). Suppose also that the prior distribution of λ is a gamma distribution with given parameters α and β ($\alpha > 0$ and $\beta > 0$). Determine the posterior distribution of λ .

Solution. Let $y = \sum_{i=1}^n x_i$. Dropping the factors which do not involve λ , we get

$$\prod_{i=1}^n f_X(x_i|\lambda) \propto e^{-n\lambda} \lambda^y.$$

Further, the prior probability density function of λ has the form

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \quad \text{for } \lambda > 0,$$

and thus it satisfies the proportionality relation

$$f_\Lambda(\lambda) \propto \lambda^{\alpha-1} e^{-\beta\lambda}.$$

Since the posterior probability density function is proportional to product of the likelihood function and the prior probability density function, it follows that

$$\left(\prod_{i=1}^n f_{\mathbf{X}}(x_i|\lambda) \right) \pi(\lambda) \propto \lambda^{\alpha+y-1} e^{-(n+\beta)\lambda} \quad \text{for } \lambda > 0.$$

We can recognize the right side of this relation, except for a constant factor, as the probability density function of a gamma distribution with parameters $\alpha + y$ and $\beta + n$. Therefore the posterior distribution of λ , given $X_i = x_i (i = 1, 2, \dots, n)$, is a gamma distribution with parameters $\alpha + \sum_{i=1}^n x_i$ and $\beta + n$. \square