Math 442: Mathematical Statistics **II**
Spring 2009
Solutions to the in-class assignments (03/04/09).

1. Suppose that $X_1, X_2, \ldots, X_n$ form a random sample from $N(\mu, 1)$ for $\mu \in [0, \infty)$. Find the maximum likelihood estimator of μ.

   Solution. For all observed values $x_1, x_2, \ldots, x_n$, the likelihood function is

$$L(\mu|\mathbf{x}) = \prod_{i=1}^{n} f_{X_i|\mu}(x_i|\mu) = \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right].$$

   We wish to maximize this function over all possible values of $\mu \geq 0$. Instead of maximizing the function directly, it is best (easier) to maximize the natural log of the function, as logarithms are increasing functions the maximum values will be the same.

$$\begin{aligned} l(\mu) &= \ln f_{\mathbf{X}|\mu}(\mathbf{x}|\mu) \\ &= \ln\left\{(2\pi)^{-n/2} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right]\right\} \\ &= -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2. \end{aligned}$$

   To locate the maximum, take the derivative with respect to μ and set it equal to zero. Solving the resulting equation will provide candidate points for optimality. In this case

$$\frac{d}{d\mu}l(\mu) = -2\sum_{i=1}^{n}(x_i-\mu) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} x_i = n\mu \quad \Rightarrow \quad \frac{1}{n}\sum_{i=1}^{n} x_i = \mu.$$

   Hence, $\mu = \overline{x}_n$ is the only candidate in the interior of the parameter space $[0, \infty)$ provided $\overline{x}_n \geq 0$. We can also see that $l'(\mu|\mathbf{x}) < 0$ if $\mu < \overline{x}_n$ and $l'(\mu|\mathbf{x}) > 0$ if $\mu > \overline{x}_n$. Therefore, if $\overline{x}_n \geq 0$, $\mu = \overline{x}_n$ is the only critical number and it is a global maximum point in the interior of the parameter space. On the other hand, if $\overline{x}_n < 0$, there is no critical number in the parameter space and hence the maximum must exist in at one of the bounders. We must also check the boundaries of the parameter space. First consider the case . Then since $l(\mu|\mathbf{x})$ is increasing for $\mu < \overline{x}_n$, we see that

$$l(0|\mathbf{x}) \leq l(\overline{x}_n|\mathbf{x})$$

   and

$$\lim_{\mu\to\infty} l(\mu|\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \exp(-\infty) = 0.$$

   Therefore, global maximum occurs at $\mu = \overline{x}_n$ if $\overline{x}_n \geq 0$. Next consider the case $\overline{x}_n < 0$. Since $l(\mu|\mathbf{x})$ is increasing for $\mu < \overline{x}_n$, we see that

$$l(0|\mathbf{x}) > l(\overline{x}_n|\mathbf{x}) > 0$$

   and

$$\lim_{\mu\to\infty} l(\mu|\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \exp[-\infty] = 0.$$

   Therefore, global maximum occurs at $\mu = 0$ if $\overline{x}_n < 0$. There for the MLE is

$$\hat{\mu} = \begin{cases} \overline{X}_n & \text{if } \overline{X}_n \geq 0 \\ 0 & \text{if } \overline{X}_n < 0 \end{cases}.$$

$\square$

2. Suppose that $X_1, X_2, \ldots, X_n$ form a random sample from a Poisson distribution for which the mean λ is known. Find the maximum likelihood estimator of $P(X = 0)$.

Solution. First we shall show that $\overline{X}_n$ is the maximum likelihood estimator of λ. For all observed values $x_1, x_2, \ldots, x_n$, the likelihood function is

$$L(\lambda|\mathbf{x}) = f_{\mathbf{X}|\lambda}(\mathbf{x}|\lambda) = \prod_{i=1}^{n} f_{X_i|\lambda}(x_i|\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}.$$

We wish to maximize this function over all possible values of $\lambda \geq 0$. Once again, it is best (easier) to maximize the the natural log of the function.

$$\begin{aligned} l(\lambda|\mathbf{x}) &= \ln L(\lambda|\mathbf{x}) \\ &= \ln \left[e^{-n\lambda} \lambda^{\left(\sum_{i=1}^{n} x_i\right)} \frac{1}{\prod_{i=1}^{n} x_i!} \right] \\ &= -n\lambda + \sum_{i=1}^{n} x_i \ln \lambda - \ln \left(\prod_{i=1}^{n} x_i! \right) \end{aligned}$$

To locate the maximum, take the derivative with respect to λ and set it equal to zero. Solving the resulting equation will provide candidate points for optimality. In this case

$$\frac{d}{d\lambda} l(\lambda) = -n + \frac{1}{\lambda} \sum_{i=1}^{n} x_i = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} x_i = n\lambda \quad \Rightarrow \quad \frac{1}{n} \sum_{i=1}^{n} x_i = \lambda.$$

Hence, $\lambda = \overline{x}_n$ is the only candidate in the interior of the parameter space. The second derivative of the function, $-(\lambda)^{-2} \sum_{i=1}^{n} x_i$, is negative at $\lambda = \overline{x}_n$. So, we have found a point of global maximum in the interior of the parameter space. We must also check the boundaries of the parameter space,

$$L(0|\mathbf{x}) = L(\infty|\mathbf{x}) = 0.$$

Thus $\lambda = \overline{x}_n$ is a global maximum. Therefore $\widehat{\lambda} = \overline{X}_n$ is the maximum likelihood estimator of λ. Then from the *invariance* property of maximum likelihood estimator, it follows that as $\overline{X}_n$ is the maximum likelihood estimator of λ, $e^{-\overline{X}_n}$ is the maximum likelihood estimator of $P(X = 0) = e^{-\lambda}$. $\square$