Mathematical Statistics **II**
Spring 2009
Solutions to the in-class assignments (2/18/09).

1. Let $X_1, X_2, \ldots$ be a sequence of random variables. By the Weak Law of Large Numbers (provided that $E(X^4) < \infty$) we have

$$\overline{X}_n \xrightarrow{p} \mu \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} {X_i}^2 \xrightarrow{p} E(X^2).$$

Use these to prove that ${S_n}^2 \xrightarrow{p} \sigma^2$ where the sample variance S_n^2 is defined by

$${S_n}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2.$$

Solution. Define $g : \mathbb{R}^2 \to \mathbb{R}$ as $g(y, z) = y - z^2$ which is a continuous function. We can now see that

$$\begin{aligned}
{S_n}^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 = \frac{1}{n}\sum_{i=1}^{n}({X_i}^2 - 2X_i\overline{X}_n + {\overline{X}_n}^2) \\
&= \frac{1}{n}\sum_{i=1}^{n} {X_i}^2 - 2\overline{X}_n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) + {\overline{X}_n}^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} {X_i}^2 - 2{\overline{X}_n}^2 + {\overline{X}_n}^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} {X_i}^2 - {\overline{X}_n}^2.
\end{aligned}$$

Now, since $\overline{X}_n \xrightarrow{p} \mu$ and $(1/n)\sum_{i=1}^{n} {X_i}^2 \xrightarrow{p} E(X^2)$, we have

$${S_n}^2 = \frac{1}{n}\sum_{i=1}^{n} {X_i}^2 - {\overline{X}_n}^2 = g\left(\tfrac{1}{n}\sum_{i=1}^{n} {X_i}^2,\ \overline{X}_n\right) \xrightarrow{p} g\left(E(X^2),\ \mu\right) = E(X^2) - \mu^2 = \sigma^2$$

as was to be shown. $\square$