Mathematical Statistics **II**
Spring 2009
Solutions to the in-class assignments (2/09/09).

1. Suppose that $(X, Y) \sim \text{trinomial}(n, (p_1, p_2))$. Find the conditional distribution of Y given $X = x$.

Solution. The joint probability density function of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{n!}{x!y!(n-x-y)!}\, {p_1}^x {p_2}^y (1-p_1-p_2)^{n-x-y} & \text{for } x, y \in \{0,1,2,\dots,n : x+y \le n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $X \sim \text{binom}(n, p_1)$, the probability density function of X is simply

$$f_X(x) = \begin{cases} \dfrac{n!}{x!(n-x)!}\, {p_1}^x (1-p_1)^{n-x} & \text{for } x = 0,1,\dots,n, \\ 0 & \text{otherwise.} \end{cases}$$

The conditional distribution of Y given $X = x$ is then

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \left(\frac{n!}{x!y!(n-x-y)!}\, {p_1}^x {p_2}^y (1-p_1-p_2)^{n-x-y} \right) \div \left(\frac{n!}{x!(n-x)!}\, {p_1}^x (1-p_1)^{n-x} \right) \\ &= \frac{n!}{x!y!(n-x-y)!} \times \frac{x!(n-x)!}{n!} \times \frac{{p_1}^x {p_2}^y (1-p_1-p_2)^{n-x-y}}{{p_1}^x (1-p_1)^{n-x}} \\ &= \frac{(n-x)!}{y!(n-x-y)!} \left(\frac{p_2}{(1-p_1)} \right)^y \left(\frac{1-p_1-p_2}{1-p_1} \right)^{n-x-y}. \end{aligned}$$

That is

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{(n-x)!}{y!(n-x-y)!} \left(\dfrac{p_2}{1-p_1} \right)^y \left(1 - \dfrac{p_2}{1-p_1} \right)^{n-x-y} & \text{for } y = 0,1,\dots,n-x, \\ 0 & \text{otherwise.} \end{cases}$$

As $p_1, p_2 > 0$ and $p_1 + p_2 \le 1$ imply $p_2 \le 1 - p_1$, it follows that $0 < \frac{p_2}{1-p_1} < 1$. Thus given that $X = x$, $Y \sim \text{Binom}\left(n - x, \frac{p_2}{1-p_1}\right)$. □

2. Suppose that the random variables $(X_1, \dots, X_{k+1})$ are independent and that $X_i \sim \text{poisson}(\lambda_i)$ for $i = 1, \dots, k+1$. Find the conditional distribution of $(X_1, \dots, X_k)$ given $\sum_{i=1}^{k+1} x_i = n$.

Solution. First note that $\sum_{i=1}^{k+1} X_i \sim$poisson$(\sum_{i=1}^{k+1} \lambda_i)$. Then for $x_i \in \{0, 1, \dots, n\}$ and $\sum_{i=1}^{k} x_i \le n$,

$$\begin{aligned}
P\left(\mathbf{X} = \mathbf{x} | \sum_{i=1}^{k+1} X_i = n\right) &= \frac{P\left(X_1 = x_1, X_2 = x_2, \dots, X_k = x_k, \sum_{i=1}^{k+1} X_i = n\right)}{P\left(\sum_{i=1}^{k+1} X_i = n\right)} \\
&= \frac{P\left(X_1 = x_1, X_2 = x_2, \dots, X_k = x_k, X_{k+1} = n - x_1 - x_2 - \cdots - x_k\right)}{P\left(\sum_{i=1}^{k+1} X_i = n\right)} \\
&= \frac{\left[\prod_{i=1}^{k} P(X_i = x_i)\right] P(X_{k+1} = n - x_1 - x_2 - \cdots - x_k)}{P\left(\sum_{i=1}^{k+1} X_i = n\right)} \\
&= \frac{\left[\prod_{i=1}^{k} \frac{e^{-\lambda_i} {\lambda_i}^{x_i}}{x_i!}\right] \frac{e^{-\lambda_{k+1}} {\lambda_{k+1}}^{(n-x_1-x_2\cdots-x_k)}}{(n - x_1 - x_2 - \cdots - x_k)!}}{\frac{e^{-\sum_{i=1}^{k+1} \lambda_i}\left[\sum_{i=1}^{k+1} \lambda_i\right]^n}{n!}} \\
&= \frac{n! \left(\prod_{i=1}^{k} \lambda_i^{x_i}\right) \lambda_{k+1}^{(n-x_1-x_2\cdots-x_k)}}{x_1! x_2! \cdots x_k! (n - \sum_{i=1}^{k} x_i)! \left[\sum_{i=1}^{k+1} \lambda_i\right]^n} \\
&= \binom{n}{x_1 \ x_2 \ \cdots x_k \ (n - \sum_{i=1}^{k} x_i)} \left[\prod_{i=1}^{k} \left(\frac{\lambda_i}{\sum_{i=1}^{k+1} \lambda_i}\right)^{x_i}\right] \left(\frac{\lambda_{k+1}}{\sum_{i=1}^{k+1} \lambda_i}\right)^{n - \sum_{i=1}^{k} x_i} \\
&= \binom{n}{x_1 \ x_2 \ \cdots x_k \ (n - \sum_{i=1}^{k} x_i)} \left[\prod_{i=1}^{k} \left(\frac{\lambda_i}{\sum_{i=1}^{k+1} \lambda_i}\right)^{x_i}\right] \\
&\quad \times \left(1 - \frac{\lambda_1}{\sum_{i=1}^{k+1} \lambda_i} - \frac{\lambda_2}{\sum_{i=1}^{k+1} \lambda_i} - \frac{\lambda_k}{\sum_{i=1}^{k+1} \lambda_i}\right)^{n - \sum_{i=1}^{k} x_i}.
\end{aligned}$$

Therefore, given that $\sum_{i=1}^{k+1} X_i = n$,

$$(X_1, X_2, \dots, X_k) \sim \text{Multinomial}(n, (\frac{\lambda_1}{\sum_{i=1}^{k+1} \lambda_i}, \frac{\lambda_2}{\sum_{i=1}^{k+1} \lambda_i}, \dots, \frac{\lambda_k}{\sum_{i=1}^{k+1} \lambda_i})).$$

□

3. Let $Z_1 \sim N(0,1)$, $Z_2 \sim N(0,1)$ and $Z_1 \perp Z_2$. For constants μ_1, μ_2, σ_1, σ_2 and ρ such that $-\infty < \mu_1, \mu_2 < \infty$, $\sigma_1, \sigma_2 > 0$ and $-1 < \rho < 1$. Define

$$\begin{aligned}
X_1 &= \sigma_1 Z_1 + \mu_1 \quad \text{and} \\
X_2 &= \sigma_2 \left[\rho Z_1 + (1 - \rho^2)^{1/2} Z_2\right] + \mu_2.
\end{aligned}$$

Show that $(X_1, X_2) \sim$ Bivariate normal$(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$. That is, show that

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi(1-\rho^2)^{1/2}\sigma_1\sigma_2} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\tfrac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\tfrac{x_1-\mu_1}{\sigma_1}\right)\left(\tfrac{x_2-\mu_2}{\sigma_2}\right) + \left(\tfrac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}.$$

Solution. The inverse transformations are

$$z_1 = \frac{x_1-\mu_1}{\sigma_1} \qquad \text{and} \qquad z_2 = \frac{x_2-\mu_2}{\sigma_2(1-\rho^2)^{1/2}} - \frac{\rho(x_1-\mu_1)}{\sigma_1(1-\rho^2)^{1/2}}.$$

The Jacobian of transformation is then

$$J = \begin{vmatrix} 1/\sigma_1 & 0 \\ -\rho/\sigma_1(1-\rho^2)^{1/2} & 1/\sigma_2(1-\rho^2)^{1/2} \end{vmatrix} = \frac{1}{\sigma_1\sigma_2(1-\rho^2)^{1/2}}.$$

The support set is unchanged. The joint distribution of (X_1, X_2) is

$$\begin{aligned}
f_{X_1,X_2}(x_1, x_2) &= \frac{1}{\sigma_1\sigma_2(1-\rho^2)^{1/2}}\, f_{Z_1,Z_2}\left(\tfrac{x_1-\mu_1}{\sigma_1}, \tfrac{x_2-\mu_2}{\sigma_2(1-\rho^2)^{1/2}} - \tfrac{\rho(x_1-\mu_1)}{\sigma_1(1-\rho^2)^{1/2}}\right) \\
&= \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2(1-\rho^2)^{1/2}} - \frac{\rho(x_1-\mu_1)}{\sigma_1(1-\rho^2)^{1/2}}\right)^2\right]\right\} \\
&= \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \\
&\quad \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\tfrac{x_1-\mu_1}{\sigma_1}\right)^2(1-\rho^2) + \left(\tfrac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\tfrac{x_1-\mu_1}{\sigma_1}\right)\left(\tfrac{x_2-\mu_2}{\sigma_2}\right) + \rho^2\left(\tfrac{x_1-\mu_1}{\sigma_1}\right)^2\right]\right\} \\
&= \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\tfrac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\tfrac{x_1-\mu_1}{\sigma_1}\right)\left(\tfrac{x_2-\mu_2}{\sigma_2}\right) + \left(\tfrac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}
\end{aligned}$$

as to be proved. □

4. Use R to plot a bivariate normal distribution with inital parameters:

$$\mu_1 = 10, \quad \mu_2 = 15, \quad {\sigma^2}_1 = 9, \quad {\sigma^2}_2 = 4, \quad \rho = 0.67.$$

Then change some parameters and observe how the plot is affected.

Solution. As can be seen in the figure below, μ_1 and μ_2 determine the centers of the elliptic contours. The parameters σ_1 and σ_2 control the spread in the x_1 and x_2 directions, respectively. The sign of ρ determines the orientation of the the ellipses. More specifically, the ellipses become slanted to the right when $\rho > 0$ and to the left when $\rho < 0$. On the other hand, the magnitude of ρ determines the length and thickness of the ellipses. In particular, when ρ is close to 1 the ellipses become longer and skinnier and when ρ is close to zero, the ellipses become more circular. □

Figure 1: Scatter Plots of Simulated Data with Contours of the underlying Bi-variate Normal pdf Superimposed