Math 442-Mathematical Statistics II
Spring 2009
Solutions to the in-class assignments (2/04/09).

1. Suppose that the random variables (X_1, X_2, X_3, X_4) are independent and $X_i \sim \text{Gamma}(\alpha_i, \beta)$ for $i = 1, 2, 3, 4$. Let,

$$U_1 = \frac{X_1}{X_1 + X_2 + X_3 + X_4}, \quad U_2 = \frac{X_2}{X_1 + X_2 + X_3 + X_4} \quad \text{and} \quad U_3 = \frac{X_3}{X_1 + X_2 + X_3 + X_4}.$$

Determine the joint distribution of (U_1, U_2, U_3).

Solution. First note that the joint distribution of (X_1, X_2, X_3, X_4) is

$$f_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4) = \begin{cases} \left(\prod_{i=1}^{4} \frac{x_i^{\alpha_i - 1}}{\Gamma(\alpha_i)\beta^{\alpha_i}}\right) e^{-\frac{1}{\beta}\sum\limits_{i=1}^{4} x_i} & \text{for } 0 < x_i < \infty \text{ and } i = 1, 2, 3, 4; \\ 0 & \text{otherwise.} \end{cases}$$

Let $U_4 = X_1 + X_2 + X_3 + X_4$, so that

$$x_1 = u_1u_4, \quad x_2 = u_2u_4, \quad x_3 = u_3u_4 \quad \text{and} \quad x_4 = u_4 - u_1u_4 - u_2u_4 - u_3u_4.$$

This implies,

$$\mathcal{B} = \{(u_1, u_2, u_3, u_4) : 0 < u_1 < 1, 0 < u_2 < 1, 0 < u_3 < 1 \quad \text{and} \quad u_4 > 0\}. \quad \text{(why?)}$$

The Jacobian of the transformation is

$$\begin{aligned} J &= \begin{vmatrix} u_4 & 0 & 0 & u_1 \\ 0 & u_4 & 0 & u_2 \\ 0 & 0 & u_4 & u_3 \\ -u_4 & -u_4 & -u_4 & (1 - u_1 - u_2 - u_3) \end{vmatrix} \\ &= u_4 \begin{vmatrix} 0 & 0 & u_1 \\ u_4 & 0 & u_2 \\ 0 & u_4 & u_3 \end{vmatrix} - u_4 \begin{vmatrix} u_4 & 0 & u_1 \\ 0 & 0 & u_2 \\ 0 & u_4 & u_3 \end{vmatrix} + u_4 \begin{vmatrix} u_4 & 0 & u_1 \\ 0 & u_4 & u_2 \\ 0 & 0 & u_3 \end{vmatrix} + (1 - u_1 - u_2 - u_3) \begin{vmatrix} u_4 & 0 & 0 \\ 0 & u_4 & 0 \\ 0 & 0 & u_4 \end{vmatrix} \\ &= {u_4}^3 u_1 + {u_4}^3 u_2 + {u_4}^3 u_3 + {u_4}^3(1 - u_1 - u_2 - u_3) = {u_4}^3. \end{aligned}$$

Then

$$\begin{aligned} f_{U_1,U_2,U_3,U_4}(u_1, u_2, u_3, u_4) &= \frac{\Gamma(\sum_{i=1}^4 \alpha_i)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)} u_1^{\alpha_1 - 1} u_2^{\alpha_2 - 1} u_3^{\alpha_3 - 1} (1 - u_1 - u_2 - u_3)^{\alpha_4 - 1} \\ &\times \frac{1}{\Gamma(\sum\limits_{i=1}^{4} \alpha_i)\beta^{\sum\limits_{i=1}^{4} \alpha_i}} u_4^{\sum\limits_{i=1}^{4} \alpha_i - 1} e^{-u_4/\beta} \quad \text{for } 0 < u_1, u_2, u_3 < 1 \text{ and } u_4 > 0 \end{aligned}$$

and $f_{U_1,U_2,U_3,U_4}(u_1, u_2, u_3, u_4) = 0$ otherwise (how?). Then clearly, $(U_1, U_2, U_3) \perp U_4$ (why?). Further,

$$f_{U_1,U_2,U_3}(u_1, u_2, u_3) = \begin{cases} \frac{\Gamma(\sum_{i=1}^4 \alpha_i)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)} u_1^{\alpha_1 - 1} u_2^{\alpha_2 - 1} u_3^{\alpha_3 - 1} (1 - u_1 - u_2 - u_3)^{\alpha_4 - 1} & \text{for } 0 < u_1, u_2, u_3 < 1 \text{ and } \sum\limits_{i=1}^{3} u_i < 1 \\ 0 & \text{otherwise} \end{cases}$$

and $U_4 \sim \mathrm{Gamma}(\sum_{i=1}^{4} \alpha_i, \beta)$. The parameters α_1, α_2, α_3, α_4 and β are all nonnegative. The random vector (U_1, U_2, U_3) is said to have *Dirichlet* distribution written as

$$(U_1, U_2, U_3) \sim \mathrm{Dirichlet}(\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

Dirichlet is a multivariate generalization of the *Beta* distribution. $\square$