Math 442-Mathematical Statistics II
Spring 2009
Solutions to the in-class assignments (2/02/09).

1. Suppose that the random variables $X_1, \ldots, X_n$ are independent and that X_i has a normal distribution with mean μ_i and variance σ_i^2 for $i = 1, \ldots, n$.

   (a) Find the distribution of $X_1 + \cdots + X_n$.

   Solution. Let $M_{X_i}(t)$ denote the moment generating function of X_i for $i = 1, \ldots, n$, and let $M_Z(t)$ denote the moment generating function of $Z = X_1 + \cdots + X_n$. Since the variables $X_1, \ldots, X_n$ are independent, then (by theorem 4.4.3)

$$\begin{aligned} M_Z(t) &= \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \exp\left(\mu_i t + \frac{1}{2}\sigma_i^2 t^2\right) \\ &= \exp\left[\left(\sum_{i=1}^{n} \mu_i\right) t + \frac{1}{2}\left(\sum_{i=1}^{n} \sigma_i^2\right) t^2\right] \quad \text{for } -\infty < t < \infty. \end{aligned}$$

   This is the moment generating function of a normal distribution. Hence, $X_1 + \cdots + X_n$ has a normal distribution with mean $\sum_{i=1}^{n} \mu_i$ and variance $\sum_{i=1}^{n} \sigma_i^2$. That is,

$$\sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right).$$

   □

   (b) Find the distribution of $\sum_{i=1}^{n}(a_i X_i + b_i)$ where $a_i, b_i \in \mathbb{R},\ i = 1, \ldots, n$.

   Solution. Note that

$$M_{a_i X_i + b_i}(t) = e^{b_i t} M_{X_i}(a_i t)$$

   for $i = 1, \ldots, n$. Let $M_Z(t)$ denote the moment generating function of $Z = \sum_{i=1}^{n}(a_i X_i + b_i)$. Since the variables $X_1, \ldots, X_n$ are independent,

$$\begin{aligned} M_Z(t) &= \prod_{i=1}^{n} e^{b_i t} M_{X_i}(a_i t) = \prod_{i=1}^{n} \exp\left((a_i \mu_i + b_i)t + \frac{1}{2}{a_i}^2 \sigma_i^2 t^2\right) \\ &= \exp\left[\left(\sum_{i=1}^{n}(a_i \mu_i + b_i)\right) t + \frac{1}{2}\left(\sum_{i=1}^{n} {a_i}^2 \sigma_i^2\right) t^2\right] \quad \text{for } -\infty < t < \infty. \end{aligned}$$

   This is the moment generating function of a normal distribution. Hence, $\sum_{i=1}^{n}(a_i X_i + b_i)$ has a normal distribution with mean $\sum_{i=1}^{n}(a_i \mu_i + b_i)$ and variance $\sum_{i=1}^{n} {a_i}^2 \sigma_i^2$. That is,

$$\sum_{i=1}^{n}(a_i X_i + b_i) \sim N\left(\sum_{i=1}^{n}(a_i \mu_i + b_i), \sum_{i=1}^{n} {a_i}^2 \sigma_i^2\right).$$

   □