Math 442-Mathematical Statistics II
Spring 2009
Solutions to the in-class assignments (1/26/09).

1. Suppose that four random variables X_1, X_2, X_3, and X_4 have a continuous joint distribution with the following joint probability density function;

$$f_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4) = \begin{cases} \dfrac{24}{(1+x_1+x_2+x_3+x_4)^5} & \text{for } x_i > 0, i = 1,2,3,4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(X_1 > X_2 > X_3 > X_4)$.

Solution.

$$P(X_1 > X_2 > X_3 > X_4) = \int \cdots \int_A f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

where $A = \{(x_1,x_2,x_3,x_4) : x_1 > x_2 > x_3 > x_4 > 0\}$. Thus,

$$\begin{aligned}
P(X_1 > X_2 > X_3 > X_4) &= \int_0^\infty \int_{x_4}^\infty \int_{x_3}^\infty \left[\int_{x_2}^\infty \frac{24}{(1+x_1+x_2+x_3+x_4)^5}\, dx_1\right] dx_2dx_3dx_4 \\
&= \int_0^\infty \int_{x_4}^\infty \int_{x_3}^\infty \left[\frac{-6}{(1+x_1+x_2+x_3+x_4)^4}\right]_{x_2}^\infty dx_2dx_3dx_4 \\
&= \int_0^\infty \int_{x_4}^\infty \left[\int_{x_3}^\infty \frac{6}{(1+2x_2+x_3+x_4)^4}\, dx_2\right] dx_3dx_4 \\
&= \int_0^\infty \int_{x_4}^\infty \left[\frac{-1}{(1+2x_2+x_3+x_4)^3}\right]_{x_3}^\infty dx_3dx_4 \\
&= \int_0^\infty \left[\int_{x_4}^\infty \frac{1}{(1+3x_3+x_4)^3}\, dx_3\right] dx_4 \\
&= \int_0^\infty \left[\frac{-1}{6(1+3x_3+x_4)^2}\right]_{x_4}^\infty dx_4 \\
&= \int_0^\infty \frac{1}{6(1+4x_4)^2}\, dx_4 \\
&= \frac{-1}{24(1+4x_4)}\Big|_0^\infty = \frac{1}{24}.
\end{aligned}$$

□

(b) Find $P(X_1 + X_2 + X_3 + X_4 > 1)$.

Solution. First compute,

$$P(X_1 + X_2 + X_3 + X_4 < 1) = \int \cdots \int_A f_{\mathbf{X}}(\mathbf{x}) \mathbf{dx}$$

where $A = \{(x_1, x_2, x_3, x_4) : 0 < x_4 < 1, 0 < x_3 < 1 - x_4, 0 < x_2 < 1 - x_3 - x_4$ and $0 < x_1 < 1 - x_2 - x_3 - x_4\}$. Thus,

$$
\begin{aligned}
&P(X_1 + X_2 + X_3 + X_4 < 1) \\
&= \int_0^1 \int_0^{1-x_4} \int_0^{1-x_3-x_4} \int_0^{1-x_2-x_3-x_4} \frac{24}{(1+x_1+x_2+x_3+x_4)^5}\, dx_1\, dx_2 dx_3 dx_4 = \frac{1}{16}
\end{aligned}
$$

The integration proceeds along the similar line as in part (a) above.

$$
\begin{aligned}
P(X_1{+}X_2 + X_3 + X_4 < 1)& \\
&= \int_0^1 \int_0^{1-x_4} \int_0^{1-x_3-x_4} \left[\int_0^{1-x_2-x_3-x_4} \frac{24}{(1+x_1+x_2+x_3+x_4)^5}\, dx_1\right] dx_2 dx_3 dx_4 \\
&= \int_0^1 \int_0^{1-x_4} \int_0^{1-x_3-x_4} \left[\frac{-6}{(1+x_1+x_2+x_3+x_4)^4}\right]_0^{1-x_2-x_3-x_4} dx_2 dx_3 dx_4 \\
&= \int_0^1 \int_0^{1-x_4} \left[\int_0^{1-x_3-x_4} \left(-\frac{3}{8} + \frac{6}{(1+x_2+x_3+x_4)^4}\right) dx_2\right] dx_3 dx_4 \\
&= \int_0^1 \int_0^{1-x_4} \left[-\frac{3}{8}x_2 + \frac{-2}{(1+x_2+x_3+x_4)^3}\right]_0^{1-x_3-x_4} dx_3 dx_4 \\
&= \int_0^1 \left[\int_0^{1-x4} \left(-\frac{3}{8}(1-x_3-x_4) - \frac{1}{4} + \frac{2}{(1+x_3+x_4)^3}\right) dx_3\right] dx_4 \\
&= \int_0^1 \left[-\frac{3}{8}x_3 + \frac{3}{16}x_3^2 + \frac{3}{8}x_3 x_4 - \frac{1}{4}x_3 + \frac{-1}{(1+x_3+x_4)^2}\right]_0^{1-x_4} dx_4 \\
&= \int_0^1 \left(-\frac{3}{16}x_4^2 + \frac{5}{8}x_4 - \frac{11}{16} + \frac{1}{(1+x_4)^2}\right) dx_4 \\
&= \left[-\frac{1}{16}x_4^3 + \frac{5}{16}x_4^2 - \frac{11}{16}x_4 + \frac{-1}{(1+x_4)}\right]_0^1 \\
&= \left(-\frac{1}{16} + \frac{5}{16} - \frac{11}{16} - \frac{1}{2} + 1\right) = \frac{1}{16}.
\end{aligned}
$$

Hence,

$$P(X_1 + X_2 + X_3 + X_4 > 1) = 1 - P(X_1 + X_2 + X_3 + X_4 > 1) = 1 - \frac{1}{16} = \frac{15}{16}.$$

$\square$