Math 442 : Solutions to Mid Term 1
Take Home Portion
Spring 2009.

Solve all problems. Show all work to receive full credit.

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with probability density function

$$f_X(x|\theta) = \begin{cases} 1/\theta & \text{if } 0 < x < \theta \text{ and } \theta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_1 = \min\{X_1, X_2, \ldots, X_n\}$ and $Y_n = \max\{X_1, X_2, \ldots, X_n\}$. Show that Y_1/Y_n and Y_n are independent.

Proof. Recall that the joint probability density functionof Y_1 and Y_n is

$$f_{Y_1,Y_n}(y_1, y_n) = n(n-1)[F_X(y_n) - F_X(y_1)]^{n-2} f_X(y_1) f_X(y_n)$$

where in this case, for $0 < t < \theta$,

$$F_X(t) = \int_0^t \frac{1}{\theta}\, dx = \frac{t}{\theta}.$$

Let $Z_1 = Y_1/Y_2$ and $Z_2 = Y_2$ so that

$$y_1 = z_1 z_2 \quad \text{and} \quad y_2 = z_2.$$

The support set for Z_1 and Z_2 is

$$\mathcal{B} = \{(z_1, z_2) : 0 < z_1 < 1,\ 0 < z_2 < \theta\}.$$

The Jacobian of transformation is then

$$J = \begin{vmatrix} z_2 & z_1 \\ 0 & 1 \end{vmatrix} = z_2.$$

Thus the joint probability density function of Z_1, Z_2 is

$$\begin{aligned} f_{Z_1,Z_2}(z_1, z_2) &= |z_2|\ f_{Y_1,Y_2}(z_1 z_2, z_2) \\ &= (z_2) n(n-1) \left[\frac{z_2}{\theta} - \frac{z_1 z_2}{\theta}\right]^{n-2} \left(\frac{1}{\theta}\right)\left(\frac{1}{\theta}\right) \\ &= (z_2) n(n-1) \left[\frac{z_2(1-z_1)}{\theta}\right]^{n-2} \left(\frac{1}{\theta^2}\right) \\ &= \frac{n(n-1)}{\theta^n}(1-z_1)^{n-2}(z_2)^{n-1}. \end{aligned}$$

Obviously, the joint pdf factors into a function of z_1 and a function of z_2 and, further, the support set of Z_1 and Z_2 are separate. Therefore, Z_1 and Z_2 are independent. That is, Y_1/Y_n and Y_n are independent. □

2. Suppose $X \sim$Exp(1). Given $X = x$, the random variables $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed as $f_{Y|X}(y|x)$ is defined by

$$f_{Y|X}(y|x) = \begin{cases} e^{-(y-x)} & \text{if } y > x, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal joint distribution of $(Y_1, Y_2, \ldots, Y_n)$.

(b) Find the conditional probability density function of X for any given value of $(Y_1, Y_2, \ldots, Y_n)$.

Solution. (a) Let $y_{\min} = \min\{y_1, y_2, \ldots, y_n\}$. Note that $f_X(x) = e^{-x}$ for $x \geq 0$. The marginal joint distribution of $\mathbf{Y}$ is

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= \int_{\mathbb{R}} f_{\mathbf{Y}|X}(\mathbf{y}|x) f_X(x) dx = \int_0^\infty \left[\prod_{i=1}^n f_{Y_i|X}(y_i|x)\right] f_X(x)\ dx = \int_0^{y_{\min}} \left[\prod_{i=1}^n e^{-(y_i - x)}\right] e^{-x}\ dx \\ &= \exp\left(-\sum_{i=1}^n y_i\right) \int_0^{y_{\min}} \exp((n-1)x)\ dx = \exp\left(-\sum_{i=1}^n y_i\right) \left[\frac{\exp((n-1)x)}{n-1}\right]_0^{y_{\min}} \\ &= \exp\left(-\sum_{i=1}^n y_i\right) \left(\frac{\exp((n-1)y_{\min}) - 1}{n-1}\right). \end{aligned}$$

That is

$$f_{\mathbf{Y}}(\mathbf{y}) = \begin{cases} \exp\left(-\sum_{i=1}^n y_i\right) \left(\dfrac{\exp((n-1)y_{\min}) - 1}{n-1}\right) & \text{if } y_{\min} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The conditional probability density function of X given $\mathbf{Y}$ is then

$$\begin{aligned} f_{X|\mathbf{Y}}(x|\mathbf{y}) &= \frac{f_{\mathbf{Y}|X}(\mathbf{y}|x) f_X(x)}{f_{\mathbf{Y}}(\mathbf{y})} \\ &= \exp\left(-\left[\sum_{i=1}^n y_i\right] + (n-1)x\right) \exp\left(\sum_{i=1}^n y_i\right) \left(\frac{n-1}{\exp((n-1)y_{\min}) - 1}\right) \\ &= \frac{(n-1)\exp((n-1)x)}{\exp((n-1)y_{\min}) - 1}. \end{aligned}$$

That is

$$f_{X|\mathbf{Y}}(x|\mathbf{y}) = \begin{cases} \dfrac{(n-1)\exp((n-1)x)}{\exp((n-1)y_{\min}) - 1} & \text{for } 0 < x < y_{\min}, \\ 0 & \text{otherwise.} \end{cases}$$

□

3. Let (X_1, X_2, X_3, X_4) have joint probability density function

$$f_{\mathbf{X}}(x_1, x_2, x_3, x_4) = \begin{cases} 24e^{-x_1 - x_2 - x_3 - x_4} & \text{if } 0 < x_1 < x_2 < x_3 < x_4 < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the transformation

$$U_1 = X_1, \quad U_2 = X_2 - X_1, \quad U_3 = X_3 - X_2 \quad \text{and} \quad U_4 = X_4 - X_3.$$

(a) Find the joint probability density function of (U_1, U_2, U_3, U_4).

(b) Find the univariate marginal distributions of U_1, U_2, U_3 and U_4.

(c) Are U_1, U_2, U_3 and U_4 mutually independent? Explain.

Soultion. (a) Inverting the equations, we have that

$$x_1 = u_1, \quad x_2 = u_2 + u_1, \quad x_3 = u_3 + u_2 + u_1 \quad \text{and} \quad x_4 = u_4 + u_3 + u_2 + u_1.$$

Since

$$0 < u_1 < u_2 + u_1 < u_3 + u_2 + u_1 < u_4 + u_3 + u_2 + u_1 < \infty,$$

the support set of (U_1, U_2, U_3, U_4) is

$$\mathcal{B} = \{(u_1, u_2, u_3, u_4) : u_1 > 0, u_2 > 0, u_3 > 0 \text{ and } u_4 > 0\}.$$

The Jacobian of transformation is then

$$J = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1$$

and so the joint probability density function of (U_1, U_2, U_3, U_4) is

$$\begin{aligned} f_{\mathbf{U}}(u_1, u_2, u_3, u_4) &= |1| f_{\mathbf{X}}(u_1, u_2 + u_1, u_3 + u_2 + u_1, u_4 + u_3 + u_2 + u_1) \\ &= 24e^{-4u_1 - 3u_2 - 2u_3 - u_4}. \end{aligned}$$

That is

$$f_{\mathbf{U}}(u_1, u_2, u_3, u_4) = \begin{cases} 24e^{-4u_1 - 3u_2 - 2u_3 - u_4} & \text{for } u_i > 0, \ i = 1, 2, 3, 4, \\ 0 & \text{otherwise.} \end{cases}$$

(b) From the joint pdf given above, we can easily see that

$$f_{U_i}(u_i) \propto e^{-(4-i+1)u_i}$$

for $u_i > 0$ and $f_{U_i}(u_i) = 0$ for $u_i \leq 0$. Since we recognize this as the kernel of Exp(β) where $\beta = 1/(4 - i + 1)$, the constant of proportionality is $(4 - i + 1)$.

(c) Since we have that

$$f_{\mathbf{U}}(u_1, u_2, u_3, u_4) = f_{U_1}(u_1) f_{U_2}(u_2) f_{U_3}(u_3) f_{U_4}(u_4) \qquad \text{for all } \mathbf{u} \in \mathcal{B},$$

by definition U_1, U_2, U_3 and U_4 are mutually independent. □

4. Let $Z_1, Z_2, \ldots$ be a sequence of random variables that converges in probability to a constant b. Assume $P(Z_n > 0) = 1$ for all n. Show that b/Z_n converges in probability.

   Proof. Let the function $g : (0, \infty) \to (0, \infty)$ be defined as $g(x) = b/x$ for a constant $b > 0$. Since $g(\cdot)$ is continuous at b, by appealing to the Continuous Mapping Theorem

   $$Z_n \xrightarrow{p} b \Rightarrow \frac{b}{Z_n} = g(Z_n) \xrightarrow{p} g(b) = \frac{b}{b} = 1.$$

   □

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with probability density function

   $$f_X(x|\theta) = \begin{cases} e^{-(x-\theta)} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

   for $\theta \in \mathbb{R}$. Let $Z_n = \min\{X_1, X_2, \ldots, X_n\}$. Prove that $Z_n \xrightarrow{p} \theta$.

Proof. For $\varepsilon > 0$,

$$\begin{aligned} P(|Z_n - \theta| > \varepsilon) = P(Z_n > \varepsilon + \theta) &= P(X_1 > \varepsilon + \theta, X_2 > \varepsilon + \theta, \dots, X_n > \varepsilon + \theta) \\ &= \prod_{i=1}^{n} P(X_i > \varepsilon + \theta) = (P(X_1 > \varepsilon + \theta))^n \end{aligned}$$

Therefore

$$\lim_{n\to\infty} P(|Z_n - \theta| > \varepsilon) = \lim_{n\to\infty} (P(X_1 > \varepsilon + \theta))^n = 0$$

since $0 < P(X_1 > \varepsilon + \theta) < 1$. Thus by definition $Z_n \xrightarrow{p} \theta$. $\square$