Math 442 : Solutions to Mid Term III
Spring 2009.

**1.** Let $X_1, \ldots, X_n$ be a random sample from

$$f(x|\theta) = \theta(1+x)^{-(1+\theta)} I_{(0,\infty)}(x), \qquad \text{for } \theta > 0.$$

(a) Find the Cramer-Rao lower bound for unbiased estimators of $\theta$.

(b) Find an efficient estimator for $1/\theta$.

**Solution:**

(a) Beginning with the log of the pdf:

$$\log f(x|\theta) = \log\theta - (1+\theta)\log(1+x).$$

Then

$$\frac{\partial}{\partial\theta}\log f(x|\theta) = \frac{1}{\theta} - \log(1+x).$$

Taking the second derivative,

$$\frac{\partial^2}{\partial\theta^2}\log f(x|\theta) = -\frac{1}{\theta^2}.$$

Therefore, the information contained in a single observation about $\theta$ is

$$I(\theta) = -\mathrm{E}\left(\frac{\partial^2 l}{\partial\theta^2}\right) = \frac{1}{\theta^2}.$$

So the CRLB is

$$\frac{1}{nI(\theta)} = \frac{\theta^2}{n}. \quad \square$$

(b) We need an estimator $T$ such that $\mathrm{E}(T) = 1/\theta = g(\theta)$ and the variance of $T$ attains the CRLB for unbiased estimators of $1/\theta$.. So we desire

$$\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f_X(x_i|\theta) = k(n,\theta)[T(x_1, x, \ldots, x_n) - g(\theta)].$$

Observe,

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f_X(x_i|\theta) &= \sum_{i=1}^{n}\frac{1}{\theta} - \log(1+x_i) \\ &= \frac{n}{\theta} - \sum_{i=1}^{n}\log(1+x_i) \\ &= -n\left[\frac{1}{n}\sum_{i=1}^{n}\log(1+x_i) - \frac{1}{\theta}\right].\end{aligned}$$

Therefore, an efficient estimator for $g(\theta) = 1/\theta$ is

$$T(X_1, X_2, \ldots, X_n) = \frac{1}{n}\sum_{i=1}^{n}\log(1+X_i). \quad \square$$

QED

**2.** Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution for which the mean is 0 and the standard deviation $\sigma$ is unknown ($\sigma > 0$). Show that the MLE of $\sigma$ given by

$$\widehat{\sigma} = \left[\frac{1}{n}\sum_{i=1}^{n}X_i^2\right]^{1/2}$$

has an asymptotic normal distribution with mean $\sigma$ and variance $\frac{\sigma^2}{2n}$.

**Solution:** We know that

$$\widehat{\sigma} \dot{\sim} \mathrm{N}\left(\sigma, \frac{1}{nI(\sigma)}\right),$$

so it suffices to show that $I(\sigma) = 2/\sigma^2$.

Recall, for a normal random variable $X$ with mean 0,

$$f_X(x|\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}.$$

Observe,

$$f(x|\sigma) = -\frac{1}{2}\log(2\pi) - \log\sigma - \frac{x^2}{2\sigma^2}$$

Then the first and second derivatives are

$$\frac{\partial}{\partial\sigma}\log f(x|\sigma) = -\frac{1}{\sigma} + \frac{x}{\sigma^3} \quad \text{and} \quad \frac{\partial^2}{\partial\sigma^2} f(x|\sigma = \frac{1}{\sigma^2} - \frac{3x^2}{\sigma^4}.$$

The information provided by a single observation about $\sigma$ is

$$I(\theta) = -\mathrm{E}\left(\frac{\partial^2 l}{\partial\theta^2}\right) \;=\; -\frac{1}{\sigma^2} + \frac{3\sigma^2}{\sigma^4} \;=\; \frac{2}{\sigma^2}.$$

Therefore,

$$\widehat{\sigma} \dot{\sim} \mathrm{N}\left(\sigma, \frac{\sigma^2}{2n}\right).$$

QED

**3.** Solve the following problems.

(a) Let $X_1, \ldots, X_n$ be a random sample from exponential($\theta$).

i. (Undergraduate Students ONLY.) Find a $100\gamma\%$ confidence interval for $\theta$.

ii. (Graduate Students ONLY.) Find a pivotal quantity based on $Y_1 = \min\{X_1, \ldots, X_n\}$ and use it to find a $100\gamma\%$ confidence interval for $1/\theta$.

(b) Let $X_1, \ldots, X_n$ be a random sample from

$$f(x|\theta) = \frac{1}{\theta} x^{(1-\theta)/\theta} I_{[0,1]}(x), \qquad \theta > 0.$$

Find a $100\gamma\%$ confidence interval for $\theta$.

**Solution:**

(a) i. We know the MGF for exponential is

$$M_X(t) = \frac{1}{1-\theta t}.$$

Consider

$$M_{\overline{X}/\theta}(t) = M_{\sum_{i=1}^n x_i}\left(\frac{t}{n\theta}\right) = \left[\frac{1}{1-(\theta t)/(n\theta)}\right]^n = \left[\frac{1}{1-t/n}\right]^n,$$

which holds by independence.

Then

$$Q = \frac{\overline{X}}{\theta} \sim \mathrm{Gamma}(n, 1/n) \implies 2nQ \sim \mathrm{Gamma}(n, 2) \overset{d}{=} \chi^2_{2n}.$$

Letting $q_1 = \chi^2_{(1-\gamma)/2, 2n}$ and $q_2 = \chi^2_{1-(1-\gamma)/2, 2n}$, we solve for a CI for $\theta$:

$$q_1 < 2nQ < q_2 \iff q_1 < \frac{2n\overline{X}}{\theta} < q_2 \iff \frac{2n\overline{X}}{q_2} < \theta < \frac{2n\overline{X}}{q_1}. \quad \diamond$$

ii. Recall for an exponential distribution,

$$f(x;\theta) = \frac{1}{\theta}e^{-x/\theta} \implies F(x;\theta) = \int_0^x \frac{1}{\theta}e^{-t/\theta}\,dt = -e^{-t/\theta}\Big|_0^x = 1 - e^{-x/\theta}.$$

Then,

$$F_{Y_1}(y;\theta) = P(Y_1 < y) = 1 - P(Y_1 \ge y) = 1 - [P(X \ge y)]^n = 1 - [e^{-y/\theta}]^n = 1 - e^{y/(\theta/n)},$$

by independence.

Thus,

$$Y_1 \sim \text{Exp}\left(\frac{\theta}{n}\right) \implies \frac{nY_1}{\theta} \sim \text{Gamma}(1,1).$$

So we know that

$$Q = \frac{2nY_1}{\theta} \sim \chi^2_2.$$

Letting $q_1 = \chi^2_{(1-\gamma)/2,2}$ and $q_2 = \chi^2_{1-(1-\gamma)/2,2}$, we solve for a CI for $\theta$:

$$\begin{aligned} & q_1 < Q < q_2 \\ \iff \quad & q_1 < \frac{2nY_1}{\theta} < q_2 \\ \iff \quad & \frac{q_1}{2nY_1} < \frac{1}{\theta} < \frac{q_2}{2nY_1}. \quad \diamond \quad \square \end{aligned}$$

(b) Observe,

$$F(x;\theta) = \int_0^x \frac{1}{\theta}t^{(1-\theta)/\theta}\,dt = \frac{1}{\theta}t^{1/\theta}\cdot\theta\Big|_0^x = x^{1/\theta}.$$

Then

$$Q = -\sum_{i=1}^n \log\left(x_i^{1/\theta}\right) \sim \text{Gamma}(n,1) \implies 2Q \sim \text{Gamma}(n,2) \stackrel{d}{=} \chi^2_{2n}.$$

Letting $q_1 = \chi^2_{(1-\gamma)/2,2n}$ and $q_2 = \chi^2_{1-(1-\gamma)/2,2n}$, we solve for a CI for $\theta$:

$$\begin{aligned} & q_1 < 2Q < q_2 \\ \iff \quad & q_1 < -\frac{2}{\theta}\sum_{i=1}^n \log x_i < q_2 \\ \iff \quad & \frac{-2\sum_{i=1}^n \log x_i}{q_1} < \theta < \frac{-2\sum_{i=1}^n \log x_i}{q_1}. \quad \square \end{aligned}$$

QED

**4.** Develop a method for estimating the parameter of the poisson distribution by a confidence interval.

**Solution:** [Note: there is more one method to do this, but only one is shown below.]

Recall the poisson pmf

$$f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}.$$

Consider a random sample from a poisson distribution $X_1, \ldots, X_n$.

The log-likelihood is given by

$$l(\lambda;\boldsymbol{x}) = \log\left(\frac{e^{-n\lambda}\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n (x_i!)}\right) = -n\lambda + \sum_{i=1}^n x_i \cdot \log\lambda - \log\left(\prod_{i=1}^n (x_i!)\right).$$

Thus, we find the MLE:

$$\frac{\partial l}{\partial \lambda} = -n + \frac{\sum_{i=1}^n x_i}{\lambda} \stackrel{\text{set}}{=} \implies \widehat{\lambda} = \overline{X}.$$

We know the mean and the variance of a poisson distribution are both $\lambda$. Then, for a large enough sample,

$$\frac{\widehat{\lambda} - \lambda}{\sqrt{\lambda/n}} \dot{\sim} \mathrm{N}(0, 1).$$

Using the MLE in the denominator we solve to find a CI for $\lambda$ is

$$\overline{X} \pm q\sqrt{\frac{\overline{X}}{n}},$$

where $q = z_{\frac{1-\gamma}{2}}$.

QED