

Math 442: Mid Term 1  
Take Home Part  
Due by: 3/02/09 at 11:00 AM  
Spring 2009.

Solve all problems. Show all your works to receive full credit.

1. Let  $X_1, X_2, \dots, X_n$  be a random sample from a population with pdf

$$f_X(x|\theta) = \begin{cases} 1/\theta & \text{if } 0 < x < \theta \quad \text{and} \quad \theta > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Let  $Y_1 = \min\{X_1, X_2, \dots, X_n\}$  and  $Y_n = \max\{X_1, X_2, \dots, X_n\}$ . Show that  $Y_1/Y_n$  and  $Y_n$  are independent.

2. Suppose  $X \sim \text{Exp}(1)$ . Given  $X = x$ , the random variables  $Y_1, Y_2, \dots, Y_n$  are iid as  $f_{Y|X}(y|x)$  defined by

$$f_{Y|X}(y|x) = \begin{cases} e^{-(y-x)} & \text{if } y > x \\ 0 & \text{otherwise} \end{cases}.$$

- (a) Find the marginal joint distribution of  $(Y_1, Y_2, \dots, Y_n)$ .  
(b) Find the conditional pdf of  $X$  for any given value of  $(Y_1, Y_2, \dots, Y_n)$ .
3. Let  $(X_1, X_2, X_3, X_4)$  have joint pdf

$$f_{\mathbf{X}}(x_1, x_2, x_3, x_4) = \begin{cases} 24e^{-x_1-x_2-x_3-x_4} & \text{if } 0 < x_1 < x_2 < x_3 < x_4 < \infty \\ 0 & \text{otherwise} \end{cases}.$$

Consider the transformation

$$U_1 = X_1, \quad U_2 = X_2 - X_1, \quad U_3 = X_3 - X_2 \quad \text{and} \quad U_4 = X_4 - X_3.$$

- (a) Find the joint pdf of  $(U_1, U_2, U_3, U_4)$ .  
(b) Find the univariate marginal distributions of  $U_1, U_2, U_3$  and  $U_4$ .  
(c) Are  $U_1, U_2, U_3$  and  $U_4$  mutually independent? Explain.
4. Let  $Z_1, Z_2, \dots$  be a sequence of random variables that converges in probability to a constant  $b$ . Assume  $P(Z_n > 0) = 1$  for all  $n$ . Show that  $b/Z_n$  converges in probability.
5. Let  $X_1, X_2, \dots, X_n$  be a random sample from a population with pdf

$$f_X(x|\theta) = \begin{cases} e^{-(x-\theta)} & \text{if } x > \theta \\ 0 & \text{otherwise} \end{cases}$$

for  $\theta \in \mathbb{R}$ . Let  $Z_n = \min\{X_1, X_2, \dots, X_n\}$ . Prove that  $Z_n \xrightarrow{p} \theta$ .