Math 442: Final Exam
Spring 2009
ONLY for graduate students.

Name ____

Solve all problems. Show all your works to receive full credit.

1. Eggs are thought to be infected with Samonella E. so that the number of organisms Y in each egg has a Poisson distribution with mean μ. The value of Y cannot be directly observed, but after some time it becomes certain whether the egg is infected ($Y > 0$) or not ($Y = 0$). Suppose that out of n eggs, r are found to be infected.

   (a) Find the maximum likelihood estimator $\widehat{\mu}$ of μ.

   (b) Find the asymptotic variance of $\widehat{\mu}$.

   (c) Is the exact variance of $\widehat{\mu}$ defined? Explain your reasoning.

2. Let $X_1, X_2, \ldots, X_n$ denote a random sample from an exponential(θ) distribution and $Y_1, Y_2, \ldots, Y_m$ denote a random sample from an exponential(β) distribution where the two samples are mutually independent.

   (a) Find a likelihood ratio test (LRT) for testing $H_0 : \theta = \beta$ versus $H_1 : \theta \neq \beta$.

   (b) Show that the LRT can be expressed in terms of the statistic

$$T = \frac{\sum\limits_{i=1}^{n} X_i}{\sum\limits_{i=1}^{n} X_i + \sum\limits_{j=1}^{m} Y_j}.$$

   (c) Argue (with out actually obtaining it) that the distribution of T does not depend on θ or β when H_0 is true (i.e. when $\theta = \beta$).

   (d) What is the distribution of T when H_0 is true? (You do not have to derive it if you know the answer.)

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from

$$f(x|\theta) = \frac{1}{\theta} x^{(1-\theta)/\theta} I_{[0,1]}(x), \quad \theta > 0.$$

   (a) Find a $100\gamma\%$ confidence interval for θ. (Hint: Find the distribution of $-\sum\limits_{i=1}^{n} \ln X_i$ as your starting place for a pivotal quantity.)

   (b) Let L be the length of the confidence interval you obtained in (3a). Find the expected value of L.

   (c) Find n such that $P(L \leq \delta\theta) \geq \rho$ for fixed δ and ρ using exact computation for $P(L \leq \delta\theta)$.

   (d) Show how Central Limit Theorem or Chebychev's Inequality can be applied to solve (3c) above. Comment on a possible limitation of these methods.

4. A general theory for the asymptotic null distribution of the likelihood ratio statistic says

$$-2\log\lambda(\mathbf{X}) \xrightarrow{\mathcal{D}} \chi^2_{p-p_0}$$

as $n \to \infty$ where χ^2_ν stands for a chi-square random variable with ν degrees of freedom, p is the number of free parameters in Ω and p_0 is the number of free parameters in Ω_0. This general theory is derived under suitable regularity conditions which includes differentiation of the pdf under the integral sign.

Now consider a sample of n observations from a uniform distribution with p.d.f.

$$f(x) = \frac{1}{2\theta}, \quad \mu - \theta \le x \le \mu + \theta.$$

It can be shown that the likelihood ratio statistic for testing $H_0 : \mu = 0$ vs $H_a : \mu \neq 0$ is

$$\lambda(\mathbf{X}) = \left(\frac{R}{2Z}\right)^n$$

where $R = X_{(n)} - X_{(1)}$, $Z = \max\{-X_{(1)}, X_{(n)}\}$, $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$.

Conduct a small-scale simulation study to investigate the following claims.

(a) The general theory on the null distribution of the likelihood ratio statistics breaks down for this particular example.

(b) The distribution χ^2_2 gives a reasonable approximation for the distribution of $-2\log\lambda(\mathbf{X})$ when H_0 is true and the sample size is large.

(c) The distribution of $-2\log\lambda(\mathbf{X})$ when H_0 is true does not depend on θ.

Explain why the general theory fails. Provide a proof to back your arguments.