Math 442 – Homework Set 5

p.461[2,5,7,11], p.496[3,7], p.505[4,10], p.513[7,14]

1. (p.461 #2) Suppose that $X_1, X_2, \ldots, X_n$ form a random sample from a uniform distribution on the interval $[0, \theta]$, and that the following hypotheses are to be tested:

$$\begin{aligned} H_0 &: \quad \theta \geq 2, \\ H_1 &: \quad \theta < 2. \end{aligned}$$

Let $Y_n = \max(X_1, X_2, \ldots, X_n)$, and consider a test procedure such that the critical region contains all the outcomes for which $Y_n \leq 1.5$.

a. Determine the power function of the test.

Solution. The power function yields the probability that Y_n is in the critical region for distributions under consideration; that is,

$$\beta(\theta) = P\left(Y_n \leq \tfrac{3}{2}\middle|\theta\right) \quad \text{for} \quad \theta \in \Omega\ .$$

We know that Y_n is the maximum likelihood estimator of θ, but more importantly if $\theta \leq \frac{3}{2}$ then it is not possible (probability zero) for X_i to be greater than $\frac{3}{2}$. Hence, if $\theta \leq \frac{3}{2}$ then $P\left(Y_n \leq \frac{3}{2}\middle|\theta\right) = 1$, as Y_n is the maximum X_i. When $\theta > \frac{3}{2}$, the probability that $Y_n \leq \frac{3}{2}$ is equivalent to the probability that every $X_i \leq \frac{3}{2}$. The probability that a single $X_i \leq \frac{3}{2}$ is $F_X(\frac{3}{2}) = \frac{3}{2}\theta$ and since $X_1, X_2, \ldots, X_n$ form a random sample (independent and identically distributed) from a uniform distribution, it follows that

$$P\left(Y_n \leq \tfrac{3}{2}\middle|\theta \geq 2\right) = \prod_{i=1}^{n} P\left(X_i \leq \tfrac{3}{2}\middle|\theta \geq 2\right) = \left(\frac{3}{2\theta}\right)^n .$$

And so,

$$\beta(\theta) = \begin{cases} 1 & \text{if } \theta \leq 3/2\ , \\ \left(\dfrac{3}{2\theta}\right)^n & \text{if } \theta > 3/2\ . \end{cases}$$

□

b. Determine the size of the test.

Solution. The size of the test is

$$\alpha = \sup_{\theta \in [2,\infty)} \beta(\theta) = \sup_{\theta \in [2,\infty)} \left(\frac{3}{2\theta}\right)^n .$$

Since θ appears in the denominator, the value of the function decreases as θ increases. It follows that the maximum is attained at $\theta = 2$. Thus the size of the test is

$$\alpha = \left(\frac{3}{4}\right)^n .$$

□

2. (p.461 #5) Suppose that $X_1, X_2, \ldots, X_n$ form a random sample from a normal distribution for which both the mean μ and the variance σ^2 are unknown. Classify each of the following hypotheses as either simple or composite:

a. $H_0: \quad \mu = 0$ and $\sigma = 1$.

b. $H_0: \quad \mu > 0$ and $\sigma < 1$.

c. $H_0: \quad \mu = -2$ and $\sigma^2 < 5$.

d. $H_0: \quad \mu = 0$.

Solution. If the statistical hypothesis completely specifies the distribution, it is called a *simple statistical hypothesis*. Otherwise it is called a *composite statistical hypothesis*. By *completely specifies the distribution* is meant, under that hypothesis the distribution from which the sample is coming is completely known, i.e. it does not contain any unknown parameter.

a. Simple. Because, the population distribution under H_0 is $N(0, 1)$ which is completely specified.

b. Composite. The population distribution under H_0 is $N(\mu, \sigma^2)$. Except that $\mu > 1$ and $\sigma < 1$, the population distribution is not completely known.

c. Composite. Under H_0, σ^2 is unknown except that $\sigma^2 < 5$. So the population distribution is not completely specified.

d. Composite. Under H_0, the population distribution is $N(0, \sigma^2)$ which is not completely specified because and σ^2 is unknown.

□

3. (p.461 #7) Return to the situation described in Example 8.1.3. Consider a different test δ^* that accepts H_0 if $2.9 \le Y_n \le 4.5$. Let δ be the test described in Example 8.1.3.

a. Prove that $\beta(\theta|\delta^*) = \beta(\theta|\delta)$ for all $\theta \le 4$.

Solution. Recall that the power function of δ is specified by the relation

$$\beta(\theta|\delta) = P(Y_n \le 2.9|\theta) + P(Y_n \ge 4|\theta) \ .$$

The power function of δ^* is specified by the relation

$$\beta(\theta|\delta^*) = P(Y_n \le 2.9|\theta) + P(Y_n \ge 4.5|\theta) \ .$$

When $\theta \le 4$, the probability that Y_n is greater than 4 is zero. Since $Y_n = \max(X_1, X_2, \ldots, X_n)$,

$$P(Y_n \ge 4.5|\theta) = P(Y_n \ge 4|\theta) = 0 \qquad \forall\ \theta \le 4 \ .$$

Therefore, for all $\theta \le 4$

$$\begin{aligned}
\beta(\theta|\delta^*) &= P(Y_n \le 2.9|\theta) + P(Y_n \ge 4.5|\theta) \\
&= P(Y_n \le 2.9|\theta) + 0 \\
&= P(Y_n \le 2.9|\theta) + P(Y_n \ge 4|\theta) \\
&= \beta(\theta|\delta).
\end{aligned}$$

That is,

$$\beta(\theta|\delta^*) = P(Y_n \le 2.9|\theta) = \beta(\theta|\delta)$$

as was to be shown. $\square$

b. Prove that $\beta(\theta|\delta^*) < \beta(\theta|\delta)$ for all $\theta > 4$.

Solution. Having defined the power functions in part (a) it suffices to show that

$$P(Y_n \ge 4.5|\theta) < P(Y_n \ge 4|\theta) \qquad \forall\ \theta > 4\ .$$

If $4 < \theta < 4.5$, then

$$P(Y_n \ge 4|\theta) = 1 - \left(\frac{4}{\theta}\right)^n$$

as was given in the example, and by reasoning similar to part (a)

$$P(Y_n \ge 4.5|\theta) = 0\ .$$

Clearly,

$$P(Y_n \ge 4.5|\theta) = 0 \quad < \quad 1 - \left(\frac{4}{\theta}\right)^n = P(Y_n \ge 4|\theta)\ .$$

Now if $\theta \ge 4.5$, then

$$P(Y_n \ge 4|\theta) = 1 - \left(\frac{4}{\theta}\right)^n$$

and

$$P(Y_n \ge 4.5|\theta) = 1 - P(Y_n \le 4.5|\theta) = 1 - \prod_{i=1}^{n} F_X(4.5|\theta) = 1 - \left(\frac{4.5}{\theta}\right)^n\ .$$

Since

$$\left(\frac{4.5}{\theta}\right)^n > \left(\frac{4}{\theta}\right)^n$$

it follows that

$$P(Y_n \ge 4.5|\theta) = 1 - \left(\frac{4.5}{\theta}\right)^n \quad < \quad 1 - \left(\frac{4}{\theta}\right)^n = P(Y_n \ge 4|\theta)\ .$$

Therefore,

$$\beta(\theta|\delta^*) < \beta(\theta|\delta) \qquad \forall\ \theta > 4$$

as was to be shown. $\square$

c. Which of the two tests seems better for testing the hypotheses (8.1.6)?

$$\begin{aligned} H_0 &: 3 \le \theta \le 4\ , \\ H_1 &: \theta < 3 \text{ or } \theta > 4\ . \end{aligned} \tag{8.1.6}$$

Solution. Both tests have the same size, $(2.9/3)^n$, and equivalent power for $\theta \le 4$. However for $\theta > 4$, since the power of the original test, δ, is greater it is better.

$\square$

4. (p.462 #11) Assume that $X_1, X_2, \ldots, X_9$ are independent and identically distributed having a Bernoulli distribution with parameter p. Suppose that we wish to test the hypotheses

$$\begin{aligned} H_0 &: \ p = 0.4, \\ H_1 &: \ p \neq 0.4. \end{aligned}$$

Let $Y = \sum_{i=1}^{9} X_i$.

a. Find c_1 and c_2 such that

$$P(Y \leq c_1 | p = 0.4) + P(Y \geq c_2 | p = 0.4)$$

is as close as possible to 0.1 without being larger than 0.1.

Solution. Being the sum of independent and identically distributed Bernoulli's, it follows that $Y \sim Binom(9, p)$. Since Y is discrete, let us compute $P(Y \leq c)$ and $P(Y \geq c)$ for $c = 0, 1, 2, \ldots, 9$.

c	$P(Y \leq c)$	$P(Y \geq c)$
0	0.01007770	1.0
1	0.07054387	0.989922304
2	0.23178701	0.929456128
3	0.48260966	0.768212992
4	0.73343232	0.517390336
5	0.90064742	0.266567680
6	0.97496525	0.099352576
7	0.99619891	0.025034752
8	0.99973786	0.003801088
9	1.0	0.000262144

It is clear that 0 and 1 are the only possible choices for c_1, while 7, 8, and 9 are the possible choices for c_2. Then perhaps it is easy to see that

$$P(Y \leq 1 | p = 0.4) + P(Y \geq 7 | p = 0.4) = 0.09557862$$

is the best option. That is, $c_1 = 1$ and $c_2 = 7$. □

b. Let δ be the test that rejects H_0 if either $Y \leq c_1$ or $Y \geq c_2$. What is the size of the test δ_c?

Solution. The power function yields the probability that Y lies within the critical region for all $p \in \Omega$, in this case

$$\beta(p|\delta_c) = P(Y \leq 1|p) + P(Y \geq 7|p) \ .$$

Since H_0 is a simple hypothesis, for $p \in \Omega_0$ the power function will take its maximum (only) value at $p = 0.4$ and as we have just shown

$$P(Y \leq 1 | p = 0.4) + P(Y \geq 7 | p = 0.4) = 0.09557862 \ .$$

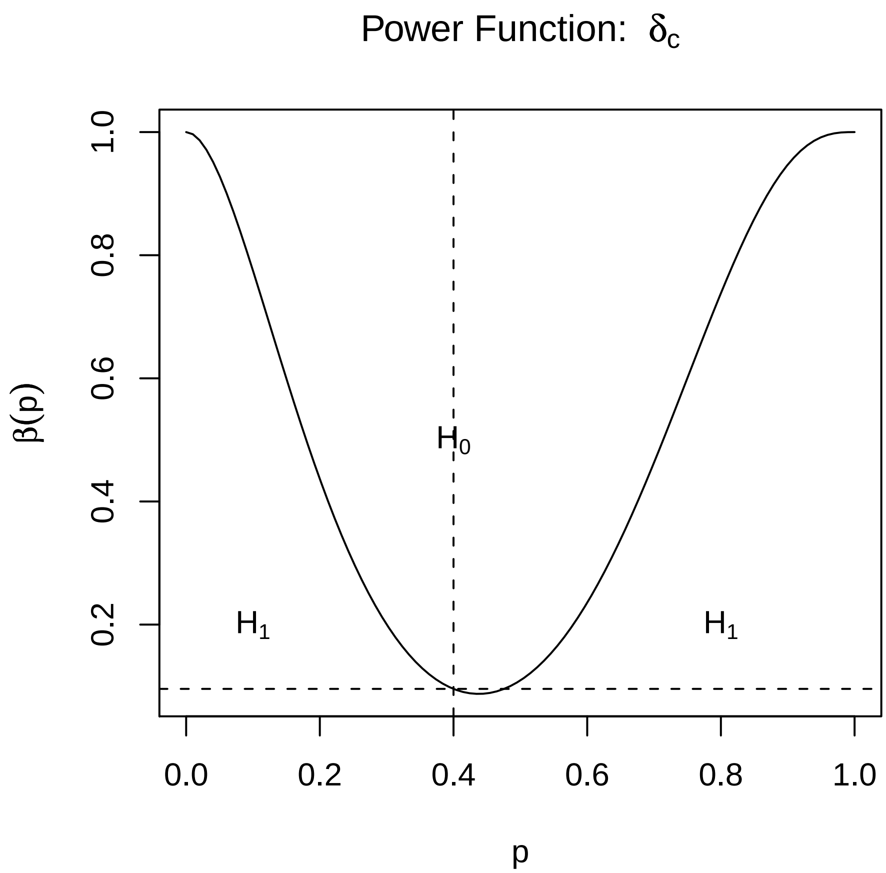
Hence, the size of the test is approximately 0.0956. □

c. Draw a graph of the power function of δ_c.

Solution. To evaluate the power function we first express it as

$$\beta(p|\delta_c) = P(Y \leq 1|p) + [1 - P(Y \leq 6|p)] \ ,$$

then it is a simple matter to plot the function over $p \in [0, 1]$ using R:



R - code:

```
# power function plot - Y ~ Binom(9, p)
# define power function
pwrfun <- function(prob){ pbinom(1, 9, prob) + (1 - pbinom(6, 9, prob)); }
# generate the data points
p <- seq(0, 1, length=100)
B <- pwrfun(p)
# plot the data
plot(p, B, type="l", main=expression(paste("Power Function:  ",delta[c])),
    xlab="p", ylab=expression(beta ( p )) )
# add lines to mark hypotheses
abline(v=0.4, lty=2)
abline(h=0.09557862, lty=2)
# label hypotheses
```

```
text(0.4,0.5, expression(H[0]))
text(0.1, 0.2, expression(H[1]))
text(0.8, 0.2, expression(H[1]))
```

□

5. (p.496 #3) The manufacturer of a certain type of automobile claims that under typical urban driving conditions the automobile will travel on average at least 20 miles per gallon of gasoline. The owner of this type of automobile notes the mileages that she has obtained in her own urban driving when she fills her automobile's tank with gasoline on nine different occasions. She finds that the results, in miles per gallon are as follows: 15.6, 18.6, 18.3, 20.1, 21.5, 18.4, 19.1, 20.4, and 19.0. Test the manufacturer's claim by carrying out a test at the level of significance $\alpha_0 = 0.05$. List carefully the assumptions you must make.

   Solution. First let us assume that the measurements are independent and identically distributed from a normal distribution with unknown mean and variance, and it is desired to test the following hypotheses:

$$\begin{aligned} H_0 &: \quad \mu \geq 20 \ , \\ H_1 &: \quad \mu < 20 \ . \end{aligned}$$

   Let the random variable X be the obtained mileage, then define

$$\overline{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{9}\sum_{i=1}^{9} x_i = 19 \ ,$$

$$s = \left(\frac{\sum_{i=1}^{n}(x_i - \overline{x}_n)^2}{n-1}\right)^{1/2} = \left(\frac{\sum_{i=1}^{9}(x_i - 19)^2}{9-1}\right)^{1/2} = \left(\frac{22}{8}\right)^{1/2} \approx 1.658312 \ ,$$

   and

$$t = \sqrt{n}\,\frac{(\overline{x}_n - \mu_0)}{s} = \sqrt{9}\,\frac{19-20}{1.658312} \approx -1.809068.$$

   So according to the t-test, H_0 will be rejected if $t < t_{0.05,8}$ Using R we can determine that

$$t_{0.05,8} = \texttt{qt(0.05, 8)} \approx -1.859548.$$

   Since $-1.809068 > -1.859548$, we do not reject the manufacturer's claim (H_0). □

6. (p. 496 #7) Consider a normal distribution for which both the mean μ and the variance σ^2 are unknown, and suppose that it is desired to test the following hypotheses:

$$\begin{aligned} H_0 &: \quad \mu \leq \mu_0, \\ H_1 &: \quad \mu > \mu_0. \end{aligned}$$

   Suppose that it is possible to observe only a single value of X from this distribution, but that an independent random sample of n observations $Y_1, Y_2, \ldots, Y_n$ is available from another normal distribution for which the variance is also σ^2, and it is known that the mean is 0. Show how to carry out a test of the hypotheses H_0 and H_1 based on the t distribution with n degrees of freedom.

Solution. Consider the maxium likelihood estimator of μ and σ, the likelihood function can be given by

$$L(\mu,\sigma|x,y_1,y_2,\dots,y_n) = \frac{1}{\sqrt{2\pi}\ \sigma}\exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}\left(\frac{1}{2\pi\ \sigma^2}\right)^{n/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}{y_i}^2\right\}.$$

The log-likelihood function is then

$$\ell(\mu,\sigma|x,y_1,y_2,\dots,y_n) = -\frac{1}{2}\ln(2\pi)-\ln\sigma-\frac{1}{2\sigma^2}(x-\mu)^2-\frac{n}{2}\ln(2\pi)-n\ln\sigma-\frac{1}{2\sigma^2}\sum_{i=1}^{n}{y_i}^2.$$

Then

$$\frac{\partial\ell}{\partial\mu}=0 \quad\Rightarrow\quad \frac{x-\mu}{\sigma^2}=0 \quad\Rightarrow\quad x=\mu$$

and

$$\frac{\partial\ell}{\partial\sigma^2}=0 \quad\Rightarrow\quad \sigma^2=\frac{1}{n+1}\sum_{i=1}^{n}y_i^2.$$

On the other hand, under Ω_0

$$\frac{\partial\ell}{\partial\mu}=\frac{1}{\sigma^2}(x-\mu).$$

Thus the likelihood function is increasing when $x>\mu$ and decreasing when $x<\mu$. Therefore the maximum of the likelihood function for each fixed σ^2 is attained at x if $x\le\mu_0$ and at μ_0 if $x>\mu_0$. Now with respect to σ^2 after profiling the mean, the maximum is attained at

$$\hat{\sigma}^2=\begin{cases}\frac{1}{n+1}\left((x-\mu_0)^2+\sum_{i=1}^{n}y_i^2\right) & \text{if } x>\mu_0\\ \frac{1}{n+1}\sum_{i=1}^{n}y_i^2 & \text{if } x\le\mu_0\end{cases}.$$

The likelihood ratio statistic is:

$$\lambda(x,y_1,\dots,y_n)=\begin{cases}\left(\frac{1}{1+\frac{1}{n}t^2}\right)^{\frac{n+1}{2}} & \text{if } x>\mu_0\\ 1 & \text{if } x\le\mu_0\end{cases}$$

where

$$t=\frac{\sqrt{n}(x-\mu_0)}{\sqrt{\sum\limits_{i=1} y_i^2/n}}.$$

The LRT rejects H_0 if

$$\lambda<c \quad\text{and}\quad x>\mu_0$$

for some $c\in(0,1)$. Equivalently, the LRT rejects H_0 if $t>k$ for some $k\in(0,\infty)$.

Let us now determine the value of k so that the test has size α. First, notice that

$$T=\frac{\sqrt{n}(X-\mu_0)}{\sqrt{\sum\limits_{i=1}^{n}Y_i^2/n}}\sim T_{n,\psi}$$

which is the noncentral t-distributions with n degrees of freedom and non-centrality parameter $\psi = \sqrt{n}(\mu - \mu_0)/\sigma$. Second, for each fixed n and k, $P(T_{n,\psi} > k)$ is an increasing function of ψ for $\psi \le 0$. Now,

$$\alpha = \sup_{\mu \le \mu_0, \sigma^2 > 0} P(T > k) = \sup_{\psi \le 0} P(T_{n,\psi} > k) = P(T_{n,0} > k).$$

That is, $k = t_{1-\alpha,n}$. Therefore, LRT rejects H_0 if $t > t_{1-\alpha,n}$.

□

7. (p. 505 #4) Suppose that $X_1, X_2, \ldots, X_m$ form a random sample from a normal distribution with mean μ_1 and variance ${\sigma_1}^2$, and $Y_1, Y_2, \ldots, Y_n$ form an independent random sample from a normal distribution with mean μ_2 and variance ${\sigma_2}^2$. Show that if $\mu_1 = \mu_2$ and ${\sigma_2}^2 = k{\sigma_1}^2$, then the random variable U defined by

$$U = \frac{(m+n-2)^{1/2}\left(\overline{X}_m - \overline{Y}_n\right)}{\left(\dfrac{1}{m} + \dfrac{k}{n}\right)^{1/2}\left(S_X^2 + \dfrac{S_Y^2}{k}\right)^{1/2}}$$

has a t distribution with $m+n-2$ degrees of freedom.

Solution. It suffices to show that

$$Z = \frac{\overline{X}_m - \overline{Y}_n}{\left(\dfrac{1}{m} + \dfrac{k}{n}\right)^{1/2}\sigma} \sim N(0,1), \quad W = \frac{S_X^2 + S_Y^2/k}{\sigma^2} \sim {\chi^2}_{m+n-2} \quad \text{and} \quad Z \perp W,$$

for we would then have that

$$U = \frac{(m+n-2)^{1/2}\left(\overline{X}_m - \overline{Y}_n\right)}{\left(\dfrac{1}{m} + \dfrac{k}{n}\right)^{1/2}\left(S_X^2 + \dfrac{S_Y^2}{k}\right)^{1/2}} = \frac{Z}{\sqrt{W/(m+n-2)}} \sim t_{m+n-2}\ .$$

Now since $\overline{X}_m$ and $\overline{Y}_n$ are independent, it follows that the difference $\overline{X}_m - \overline{Y}_n$ has a normal distribution with mean $\mu_1 - \mu_2$ and variance

$$\frac{\sigma^2}{m} + \frac{k\sigma^2}{n} = \left(\frac{1}{m} + \frac{k}{n}\right)\sigma^2 \qquad \text{where } \sigma^2 = {\sigma_1}^2\ .$$

When $\mu_1 = \mu_2$ the mean of the difference, $\overline{X}_m - \overline{Y}_n$, will be zero. Therefore

$$Z = \frac{\overline{X}_m - \overline{Y}_n}{\left(\dfrac{1}{m} + \dfrac{k}{n}\right)^{1/2}\sigma} \sim N(0,1)$$

as this is a normal distribution minus its mean and divided by its standard deviation.

Having defined

$$S_X^2 = \sum_{i=1}^{m}(X_i - \overline{X}_m)^2 \quad \text{and} \quad S_Y^2 = \sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2$$

it follows that
$$\frac{S_X^2}{\sigma^2} \sim {\chi^2}_{m-1} \quad \text{and} \quad \frac{S_Y^2}{k\sigma^2} \sim {\chi^2}_{n-1} \qquad \text{where } \sigma^2 = {\sigma_1}^2$$
as they are the sum squares of m and n standard normals, respectfully. Hence,
$$W = \frac{S_X^2 + S_Y^2/k}{\sigma^2} \sim {\chi^2}_{m+n-2}.$$
Finally, since $(\overline{X}, S_X^2) \perp (\overline{Y}, S_Y^2)$ and the two samples are independent, we have that $(\overline{X} - \overline{Y}) \perp (S_X^2, S_Y^2)$. Consequently, $Z \perp W$. In conclusion, U has a t distribution with $m+n-2$ degrees of freedom. □

8. (p.505 #10) Lyle et al. (1987) ran an experiment to study the effect of a calcium supplement on the blood pressure of African-American males. A group of 10 men received a calcium supplement, and another group of 11 men received a placebo. The experiment lasted 12 weeks. Both before and after the 12-week period, each man had his systolic blood pressure measured while at rest. The changes (after minus before) are given in Table 1. Test the null hypothesis that the mean change in blood pressure for the calcium supplement group is lower than the mean change in blood pressure for the placebo group. Use level $\alpha_0 = 0.1$.

Calcium	7	-4	18	17	-3	-5	1	10	11	-2	
Placebo	-1	12	-1	-3	3	-5	5	2	-11	-1	-3

Table 1: Blood pressure data for Question 8.

Solution. Let the random variable X has been observed to get the measurements from the calcium group and Y has been observed to get the measurements in the placebo group. Assume $X \sim N(\mu_C, \sigma_C^2)$, $Y \sim N(\mu_P, \sigma_P^2)$ and $X \perp Y$. The hypotheses of interest are:
$$H_0 : \mu_C \leq \mu_P \quad \text{vs} \quad H_1 : \mu_C > \mu_P.$$
We will use R to analyze this data under equal and unequal variances. Let us first enter the data.

```
Calcium=c(7,-4,18,17,-3,-5,1,10,11,-2)
Placebo=c(-1,12,-1,-3,3,-5,5,2,-11,-1,-3)
```

Assuming equal variances, we get the following result.

```
> t.test(Calcium,Placebo,alternative="greater", conf.level=0.90, var.equal = TRUE)

        Two Sample t-test

data:  Calcium and Placebo
t = 1.6341, df = 19, p-value = 0.05935
alternative hypothesis: true difference in means is greater than 0
90 percent confidence interval:
 0.9885871       Inf
```

```
sample estimates:
 mean of x  mean of y
 5.0000000 -0.2727273
```

Since p-value is less than 0.1, we reject H_0 and conclude that the mean change in blood pressure for calcium supplement group is higher than the mean change in blood pressure for the placebo group.

Next we do the analysis without the assumption of equality of variances.

```
> t.test(Calcium,Placebo,alternative="greater",conf.level=0.90, var.equal = FALSE)

        Welch Two Sample t-test

data:  Calcium and Placebo
t = 1.6037, df = 15.591, p-value = 0.06442
alternative hypothesis: true difference in means is greater than 0
90 percent confidence interval:
 0.8727392       Inf
sample estimates:
 mean of x  mean of y
 5.0000000 -0.2727273
```

We see that the decision will be the same. Hence, for this problem the assumption of equality of variance is immaterial as far as the decision goes. □

9. (p.513 #7) Consider two different normal distributions for which the both the means μ_1 and μ_2 and the variances ${\sigma_1}^2$ and ${\sigma_2}^2$ are unknown, and suppose that it is desired to test the following hypotheses:

$$H_0: \quad {\sigma_1}^2 \leq {\sigma_2}^2,$$
$$H_1: \quad {\sigma_1}^2 > {\sigma_2}^2.$$

Suppose further that a random sample consisting of 16 observations for the first normal distribution yields the values $\sum_{i=1}^{16} X_i = 84$ and $\sum_{i=1}^{16} {X_i}^2 = 563$, and an independent random sample consisting of 10 observations from the second normal distribution yields the values $\sum_{i=1}^{10} Y_i = 18$ and $\sum_{i=1}^{10} Y_i^2 = 72$.

a. What are the maximum likelihood estimator's of ${\sigma_1}^2$ and ${\sigma_2}^2$?

b. If an F test is carried out at the level of significance 0.05, is the hypothesis H_0 accepted or rejected?

10. (p.514 #14) Consider again the conditions of Exercise 7 [Question 9]. Find the power function of the F test when ${\sigma_1}^2 = 2{\sigma_2}^2$.