Math 442 – Homework Set 3

p.291[9,11,12,14], p.294[1], p.396[1,2,8], p.404[7,8,9], p.409[3], p.446[5,8,9]

1. (p.291 #9) A physicist makes 25 independent measurements of the specific gravity of a certain body. He knows that the limitations of his equipment are such that the standard deviation of each measurement is σ units.

   a. By using the Chebyshev inequality, find a lower bound for the probability that the average of his measurements will differ from the actual specific gravity of the body by less than $\sigma/4$ units.

   b. By using the central limit theorem , find an approximate value for the probability in part (a).

   Solution. As the measurements form a random sample from a normal distribution with mean μ and variance σ^2, the average of the physicist's measurements will have a normal distributed with mean $E(\overline{X}_n) = \mu$ and variance $\mathrm{Var}(\overline{X}_n) = \sigma^2/25$.

   a. Using Chebyshev's inequality (reversing the inequality inside the probability)

   $$P\left(|\overline{X}_n - E(X)| \le \frac{\sigma}{4}\right) \ge 1 - \frac{\sigma^2/25}{\sigma^2/16} = 1 - \frac{16}{25} = \frac{9}{25} = 0.36.$$

   Using this method, the probability that the average of the physicist's measurements is within $\sigma/4$ units of the actual specific gravity is at least 0.36.

   b. Using the central limit theorem,

   $$\begin{aligned} P\left(|\overline{X}_n - \mu| \le \frac{\sigma}{4}\right) = \quad & P\left(-\frac{\sigma}{4}\cdot\frac{5}{\sigma} \le \frac{5}{\sigma}\left[\overline{X}_n - \mu\right] \le \frac{\sigma}{4}\cdot\frac{5}{\sigma}\right) \\ &= P\left(\frac{5(\overline{X}_n - \mu)}{\sigma} \le \frac{5}{4}\right) - P\left(\frac{5(\overline{X}_n - \mu)}{\sigma} \le -\frac{5}{4}\right) \\ &\approx \Phi(5/4) - \Phi(-5/4) = 0.7887005 \end{aligned}$$

   Using this method, the probability that the average of the physicist's measurements is within $\sigma/4$ units of the actual specific gravity is 0.7888, a way more accurate than the the lower bound estimate from Chebyshev's inequality found in part (a).

   □

2. (p.291 #11) Suppose that, on the average, one third of the graduating seniors at a certain college have two parents attend the graduation ceremony, another third of these seniors have one parent attend the ceremony , and the remaining third of these seniors have no parents attend. If there are 600 graduating seniors in a particular class, what is the probability that not more than 650 parents will attend the graduation ceremony?

   Solution. Let X_i be the number of parents which attend for the ith student. Then

   $$E(X_i) = (0)\frac{1}{3} + (1)\frac{1}{3} + (2)\frac{1}{3} = 1 \quad \text{and} \quad \mathrm{Var}(X_i) = (0)^2\frac{1}{3} + (1)^2\frac{1}{3} + (2)^2\frac{1}{3} - 1 = \frac{2}{3}.$$

Taking a random sample of 600 students we wish to determine the probability that $\sum_{i=1}^{600} X_i$ is less than or equal to 650 ("not more"), in this case the central limit theorem can provide us with a good approximation.

$$\begin{aligned}
P\left(\sum_{i=1}^{600} X_i \leq 650\right) &= P(600\overline{X}_n \leq 650) \\
&= P\left(\frac{600\overline{X}_n - 600}{\sqrt{2/3}} \leq \frac{650-600}{\sqrt{2/3}}\right) \\
&= P\left(\frac{\sqrt{600}(\overline{X}_n - 1)}{\sqrt{2/3}} \leq \frac{650-600}{\sqrt{600}\sqrt{2/3}}\right) \\
&= P\left(\frac{\sqrt{600}(\overline{X}_n - 1)}{\sqrt{2/3}} \leq \frac{5}{2}\right) \\
&\approx \Phi(5/2) = \texttt{0.9937903}
\end{aligned}$$

Thus with 600 students, it is very unlikely that more that 650 parents will attend the ceremony. □

3. (p.291 #12) Let X_n be a random variable having a binomial distribution with parameters n and p_n. Assume that $\lim_{n\to\infty} np_n = \lambda$. Prove that the moment generating function of X_n converges to the moment generating function of a Poisson distribution with mean λ.

   Solution. Here we shall make use of the fact that

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n,$$

   or as stated in our book. More generally, it is shown in advanced calculus classes that if a sequence a_n converges to b (i.e. $\lim_{n\to\infty} a_n = b$), then

$$\lim_{n\to\infty}\left(1+\frac{a_n}{n}\right)^n = e^b.$$

   Let M_{X_n} denote the moment generating function of X_n, then by definition

$$\lim_{n\to\infty} M_{X_n}(t) = \lim_{n\to\infty}\left(p_n e^t + 1 - p_n\right)^n = \lim_{n\to\infty}\left(1+\frac{np_n(e^t-1)}{n}\right)$$

   and from our assumption that $\lim_{n\to\infty} np_n = \lambda$ it follows that

$$\lim_{n\to\infty}\left(1+\frac{np_n(e^t-1)}{n}\right) = \exp[\lambda(e^t-1)] \text{ for all } t.$$

   Apparently, $\exp[\lambda(e^t-1)]$ is the moment generating function of a Poisson distribution with mean λ, as was to be shown. □

4. (p.291 #14) Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with mean 0 and unknown variance σ^2.

   a. Determine the asymptotic distribution of the statistic $\left(\frac{1}{n}\sum_{i=1}^n X_i{}^2\right)^{-1}$.

   b. Find a variance stabilizing transformation for the statistic $\frac{1}{n}\sum_{i=1}^n X_i{}^2$.

Solution. a. Let $Y_i = X_i^2$ then as $\mu = 0$,

$$E(Y_i) = E(X_i^2) = E(X_i^2) - \mu^2 = \sigma^2 \quad \text{and} \quad \text{Var}(Y_i) = \text{Var}(X_i^2) = E(X_i^4) - \sigma^4.$$

The expected value of X_i^4 can be determined using the moment generating function. That is,

$$\begin{aligned} E(X_i^4) &= \tfrac{\partial^4}{\partial^4 t} M_{X_i}(t)|_{t=0} = \tfrac{\partial^4}{\partial^4 t} \exp\left[-\frac{1}{2}\,\sigma^2 t^2\right]\,|_{t=0} \\ &= \left[\sigma^8 t^4 e^{-\frac{\sigma^2 t^2}{2}} - 6\sigma^6 t^2 e^{-\frac{\sigma^2 t^2}{2}} + 3\sigma^4 e^{-\frac{\sigma^2 t^2}{2}}\right]_{t=0} \\ &= 3\sigma^4 \end{aligned}$$

hence, $\text{Var}(Y_i) = 2\sigma^4$. By the CLT, it follows that

$$\overline{Y}_n = \frac{1}{n}\sum_{i=1}^n X_i^2 \sim N\left(\sigma^2, \frac{2\sigma^4}{n}\right).$$

Here the given statistic is a function of $\overline{Y}_n$, so to determine its asymptotic distribution we apply the delta method. Let $g(x) = 1/x$, the function is differentiable and $g'(\sigma^2) = -1/\sigma^4 \neq 0$ for $\sigma^2 > 0$. Then by virtue of the delta method, the asymptotic distribution of $g(\overline{Y}_n)$ is

$$\left(\frac{1}{n}\sum_{i=1}^n X_i^2\right)^{-1} \sim N\left(-\frac{1}{\sigma^4}(\sigma^2) + \frac{2}{\sigma^2}, \frac{1}{\sigma^8}\left(\frac{2\sigma^4}{n}\right)\right) = N\left(\frac{1}{\sigma^2}, \frac{2}{n\sigma^4}\right).$$

b. In part (a) we showed that

$$\frac{1}{n}\sum_{i=1}^n X_i^2 \sim N\left(\sigma^2, \frac{2\sigma^4}{n}\right),$$

here the variance is a function of the mean. So the variance stabilizing transformation, g, shall have the form

$$g(\sigma^2) = \int_1^{\sigma^2} \frac{1}{\sqrt{2x^2/n}}\,dx = \sqrt{n/2}\int_1^{\sigma^2} \frac{1}{x}\,dx = \sqrt{n/2}\,\ln\sigma^2.$$

□

5. (p.294 #1) Let $X_1, \ldots, X_{30}$ be independent random variables each having a discrete distribution with probability function

$$f_X(x) = \begin{cases} 1/4 & \text{if } x = 0 \text{ or } 2, \\ 1/2 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Use the central limit theorem and the correction for continuity to approximate the probability that $X_1 + \cdots + X_{30}$ is at most 33.

Solution. First compute the expected value and variance of X_i,

$$E(X_i) = (0)\frac{1}{4} + (2)\frac{1}{4} + (1)\frac{1}{2} = 1 \quad \text{and} \quad \text{Var}(X_i) = (0)^2\frac{1}{4} - (2)^2\frac{1}{4} - (1)^2\frac{1}{2} - 1 = \frac{1}{2}.$$

Adding a correction for continuity, we wish to determine the probability that $\sum_{i=1}^{30} X_i$ is less than or equal to 33.5. By applying the central limit theorem, we are able to determine that

$$P\left(\sum_{i=1}^{30} X_i \leq 33.5\right) = P\left(\frac{\sqrt{30}(\overline{X}_n - 1)}{\sqrt{1/2}} \leq \frac{33.5 - 30}{\sqrt{30}\sqrt{1/2}}\right) \approx \Phi(0.9036961) = \texttt{0.8169217}.$$

Whereas if we had not added in the correction for continuity the result would have been `0.780711`.

□

6. (p.396 #1) Suppose that we will sample 20 chunks of cheese, as described in Example 7.2.1. Let $T = \sum_{i=1}^{20}(X_i - \mu)^2/20$, where X_i is the concentration of lactic acid in the ith chunk. Assume that $\sigma^2 = 0.09$. What number c satisfies $P(T \leq c) = 0.9$?

Solution. We can write T as

$$T = \sum_{i=1}^{20} \frac{(X_i - \mu)^2}{0.09(222.222\ldots)} = \frac{9}{2000}\sum_{i=1}^{20}\left(\frac{X_i - \mu}{\sqrt{0.09}}\right)^2.$$

Then, since each term in the sum is the square of a standard normal, it follows that $\frac{2000}{9}T$ has a χ^2 distribution with 20 degrees of freedom. Using a quantile table or the quantile function in R (`qchisq(0.90, 20)`) we can determine that

$$P\left(\frac{2000}{9}T \leq 28.41198\right) = 0.90 \quad \Rightarrow \quad c = \frac{9}{2000}(28.41198) = 0.1278539.$$

□

7. (p.396 #2) Find the mode of the χ^2 distribution with ν degrees of freedom ($\nu = 1, 2, \ldots$).

Solution. First note that a *mode* of a distribution (of a random variable X) is a value of x that maximizes the p.d.f. $f_X(x)$. For X of the continuous type, $f_X(x)$ is a continuous function. If there is only one such x, it is called the *mode of the distribution*. The probability density function of the χ^2 distribution with ν degrees of freedom is given by

$$f_X(x|\nu) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2)-1}e^{-x/2}.$$

To find the value of x which maximizes this function we apply some calculus. Let ν be a constant in $\mathbb{Z}^+$, then with ν fixed maximizing $f_{X|\nu}$ is equivalent to maximizing

$$g(x) = \ln\left[x^{(\nu/2)-1}e^{-x/2}\right] = [(\nu/2) - 1]\ln x - x/2$$

as we can ignore the constant and taking the natural log will preserve the maximum value(s). To find the maximum take the derivative with respect to x and set it equal 0, solving for x yields the candidate point(s). In this case

$$\frac{d}{dx}g(x) = \frac{(\nu/2) - 1}{x} - \frac{1}{2} = 0 \quad \Rightarrow \quad \nu - 2 = x$$

hence $\widehat{x} = \nu - 2$ for $\nu > 2$ and there is no critical number for $\nu \leq 2$ because the support set of χ^2_ν is $(0, \infty)$. Having found a candidate, $\widehat{x}$, we must confirm that is the global maximum. The second derivative, $-(x)^{-2}[(n/2) - 1]$, is negative (this implies that the function is convex) at $\widehat{x}$

so we have found a maximum. We must also also check the endpoints of the support set, that is, the limits of the original function,

$$\lim_{x\to\infty} \frac{1}{2^{\nu/2}\Gamma(\nu/2)}\ x^{(\nu/2)-1}e^{-x/2} = 0,$$

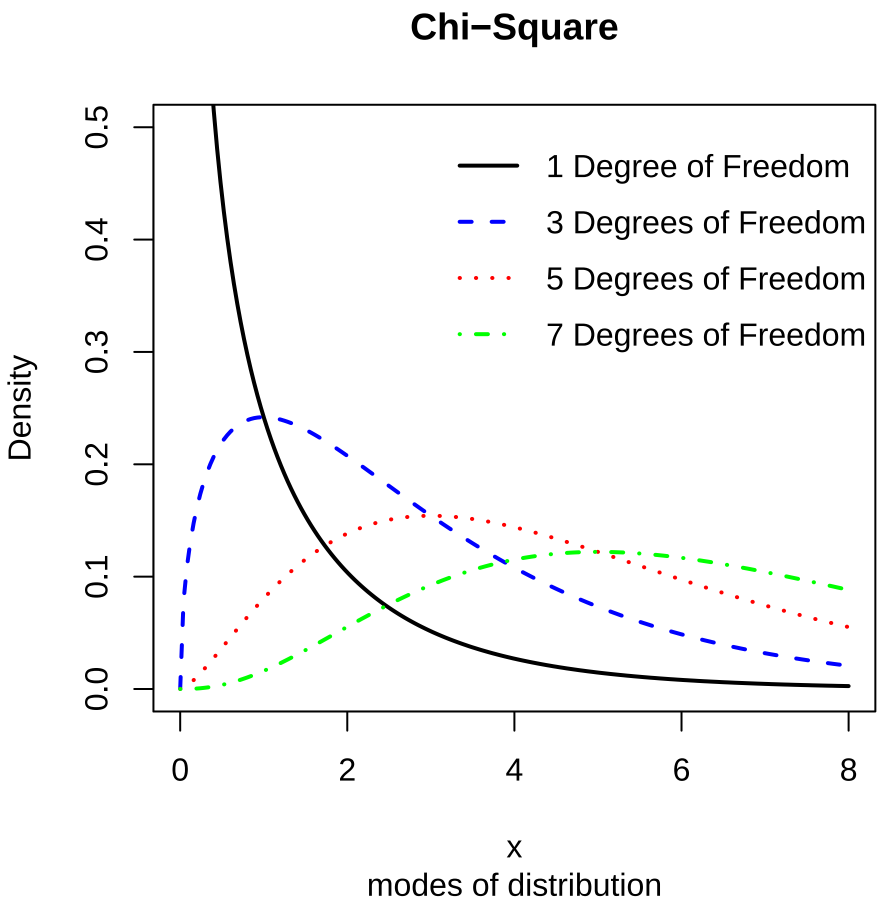
and

$$\lim_{x\to 0} \frac{1}{2^{\nu/2}\Gamma(\nu/2)}\ x^{(\nu/2)-1}e^{-x/2} = 0 \quad \text{when } (\nu/2)-1>0,$$

but

$$\lim_{x\to 0} \frac{1}{2^{\nu/2}\Gamma(\nu/2)}\ x^{(\nu/2)-1}e^{-x/2} = \infty \quad \text{when } (\nu/2)-1\le 0.$$

Thus $\widehat{x}$ is a global maximum for $\nu > 2$ and therefore the mode of the distribution is $\nu - 2$ for $\nu > 2$. For $\nu = 1, 2$ the mode of the distribution does not exist. See the figure below. □



8. (p.396 #8) Suppose that $X_1, \ldots, X_n$ form a random sample from a uniform distribution on the interval $[0, 1]$, and let W denote the range of the sample, as defined in Section 3.9. Also let $Z = 2n(1 - W)$ with the probability density function of Z denoted as $f_Z(z|n)$ and let f_{χ^2} denote the probability density function of the χ^2 distribution with four degrees of freedom. Show that

$$\lim_{n\to\infty} f_Z(z|n) = f_{\chi^2}(z) \quad \text{for } z > 0.$$

Proof. Recall that, the probability density function of W is

$$f_W(w|n) = \begin{cases} n(n-1)w^{n-2}(1-w) & \text{for } w \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = 1 - W$, then

$$f_Y(y|n) = |-1| f_W(1-y|n) = \begin{cases} n(n-1)(1-y)^{n-2}(y) & \text{for } y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

The probability density function of $Z = 2nY$ is then

$$f_Z(z|n) = \left|\frac{1}{2n}\right| f_Y\left(\frac{z}{2n}\middle| n\right) = \frac{1}{2n} n(n-1)\left(1 - \frac{z}{2n}\right)^{n-2}\left(\frac{z}{2n}\right) \quad \text{for } 0 < z < 2n.$$

All that is required now is some algebra,

$$\lim_{n\to\infty} f_Z(z|n) = \lim_{n\to\infty} \frac{1}{4}\left(\frac{n-1}{n}\right) z \left(1 + \frac{-z/2}{n}\right)^{n-2}$$

then as

$$\lim_{n\to\infty} \frac{n-1}{n} = 1 \qquad \text{and} \qquad \lim_{n\to\infty}\left(1 + \frac{-z/2}{n}\right)^{n-2} = e^{-z/2}$$

we have that

$$\lim_{n\to\infty} f_Z(z|n) = \frac{1}{4} ze^{-z/2} = \frac{1}{2^2\ 1!} z^{2-1}e^{-z/2} = \frac{1}{2^{4/2}\ \Gamma(4/2)} z^{(4/2)-1}e^{-z/2},$$

which is the probability density function of f_{χ^2}, a χ^2 distribution with 4 degrees of freedom, as was to be shown. □

9. (p. 404 #7) Suppose that $X_1, \ldots X_n$ form a random sample from a normal distribution with mean μ and variance σ^2, and let $\widehat{\sigma^2}$ denote the sample variance. Determine the smallest values of n for which the following relations are satisfied:

   a. $P\left(\frac{\widehat{\sigma^2}}{\sigma^2} \le 1.5\right) \ge 0.95$;

   b. $P\left(|\widehat{\sigma^2} - \sigma^2| \le \frac{1}{2}\sigma^2\right) \ge 0.8$.

Solution. a. We begin by converting the probability in terms of a χ^2 distribution with $n-1$ degrees of freedom (Theorem 7.3.1). Let $V \sim {\chi^2}_{n-1}$. Then

$$P\left(\frac{\widehat{\sigma^2}}{\sigma^2} \le 1.5\right) = P\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i - \overline{X}_n}{\sigma}\right)^2 \le 1.5\right) = P(V \le 1.5n).$$

One could use the table in the back of the book to determine the minimum value of n for which $P(V \le 1.5n) \ge 0.95$, however I prefer the more modern approach. Using R to get an approximate (integer) solution, the result was 21.

```
> n <- seq(1,50,by=1) # sequence of integers, could start at larger value
> n[1] = 2 # ensure degrees of freedom stays above zero, while preserving index
> prob <- pchisq(1.5*n, df=n-1)
> # c.d.f. of chi-sq  P(X <= 1.5*n), degrees of freedom passed in as vector
> which(prob >= 0.95)[1] # find the minimum value of n, for which prob >= 0.95
[1] 21
```

b. Once again, we begin by converting the probability into terms of a χ^2 distribution with $n-1$ degrees of freedom. Let $V \sim {\chi^2}_{n-1}$

$$\begin{aligned} P\left(|\widehat{\sigma^2} - \sigma^2| \le \frac{1}{2}\sigma^2\right) &= P\left(-\frac{1}{2}\sigma^2 \le \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 - \sigma^2 \le \frac{1}{2}\sigma^2\right) \\ &= P\left(-\frac{1}{2} \le \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i - \overline{X}_n}{\sigma}\right)^2 - 1 \le \frac{1}{2}\right) \\ &= P\left(\frac{1}{2}\,n \le V \le \frac{3}{2}\,n\right) \\ &= P(V \le 3n/2) - P(V \le n/2) \end{aligned}$$

Using R, the smallest value of n for which $P(V \le 3n/2) - P(V \le n/2) \ge 0.8$ is 13.

```
> n <- seq(1,50,by=1) # sequence of integers, could start at larger value
> n[1] = 2 # ensure degrees of freedom stays above zero
> prob <- pchisq(1.5*n, df=n-1)-pchisq(0.5*n, df=n-1)
> # c.d.f. of chi-sq  P(0.5*n <= X <= 1.5*n), degrees of freedom passed in as vector
> which(prob >= 0.80)[1] # find the minimum n for which prob >= 0.80
[1] 13
```

□

10. (p.404 #8) Suppose that X has a χ^2 distribution with 200 degrees of freedom. Explain why the central limit theorem can be used to determine the approximate value of $P(160 < X < 240)$ and find this approximate value.

Solution. Let $Y_1, Y_2, \ldots, Y_{200}$ be independent and identically distributed random variables each having a χ^2 distribution with one degree of freedom, then

$$X = \sum_{i=1}^{200} Y_i \qquad \text{that is,} \qquad \sum_{i=1}^{200} Y_i \sim {\chi^2}_{200}.$$

The central limit theorem tell us that for large n (200 is large)

$$P\left[\frac{n^{1/2}(\overline{Y}_n - \mu)}{\sigma} \le y\right] \approx \Phi(y).$$

In this case $\mu = 1$ and $\sigma = \sqrt{2}$, as Y_i has a χ^2 distribution with one degree of freedom. Here the sample mean, $\overline{Y}_n$, is $\frac{1}{200}\sum_{i=1}^{200} Y_i$ which is equivalent to $X/200$. Thus the distribution function of X can be approximated using the central limit theorem, as

$$P\left[\frac{10\sqrt{2}(X/200 - 1)}{\sqrt{2}} \le x\right] = P\left[X \le 20x + 200\right] \approx \Phi(x).$$

It follows that

$$\begin{aligned} P(160 < X < 240) &= P(X < 240) - P(X < 160) \\ &= P\left(X < 20(2) + 200\right) - P\left(X < 20(-2) + 200\right) \\ &\approx \Phi(2) - \Phi(-2) = 2\Phi(2) - 1 = 0.9546 \qquad \text{using a table.} \end{aligned}$$

Now let us find the exact probaility using chi^2 distribution. Using R,

```
> pnorm(2,mean=0,sd=1)-pnorm(-2,mean=0,sd=1)
[1] 0.9544997
> pchisq(240,df=200)-pchisq(160,df=200)
[1] 0.955028
```

In conclusion, the central limit theorem is usefully in this case because the random variable X can be expressed as the sum of independent and identically distributed random variables have a known mean and variance, in other words X can be expressed as a function of the sample mean. □

11. (p.404 #9) Suppose that each of two statisticians A and B independently takes a random sample of 20 observations from a normal distribution for which the mean μ is unknown and the value of the variance is 4. Suppose also that statistician A finds the sample variance in his random sample to be 3.8, and statistician B finds the sample variance in her random sample to be 9.4. For which random sample is the sample mean likely to be closer to the unknown value of μ?

    Solution. We know that the sample mean is independent of the sample variance. Therefore, no information about the sample mean is provided by the sample variance. In other words, knowing the sample variances do not tell us any information about the sample means. Further, since the two samples are taken from the same population independently, before observing the samples we know that $\overline{X}_A$ and $\overline{X}_B$ have the same distribution. So we expect the observed sample means to be equally likely to be close to the unknown population mean μ.

    □

12. (p.409 #3) Suppose that the five random variables $X_1, \ldots, X_5$ are independent and identically distributed, and each has a standard normal distribution. Determine a constant c such that the random variable

$$\frac{c(X_1 + X_2)}{\left(X_3{}^2 + X_4{}^2 + X_5{}^2\right)^{1/2}}$$

will have a t distribution.

Solution. Recall that the t distribution is defined as follows. Consider two independent random variables Y and Z, such that Y has a χ^2 distribution with ν degrees of freedom and Z has a standard normal distribution. Suppose that a random variable X is defined by the equation

$$X = \frac{Z}{\left(\dfrac{Y}{\nu}\right)^{1/2}}.$$

Then the distribution of X is called the t *distribution with* ν *degrees of freedom.*

Letting

$$Z = c_1(X_1 + X_2) \quad \text{and} \quad \left(\frac{Y}{\nu}\right)^{1/2} = c_2\left(X_3{}^2 + X_4{}^2 + X_5{}^2\right)^{1/2}$$

we wish to find a $c = c_1/c_2$ such that $Z \sim N(0,1)$ and $Y \sim \chi^2_\nu$. To that end, first consider $c_1(X_1 + X_2)$, we know that the sum of these two normal random variables shall have a normal distribution with mean

$$E(c_1X_1 + c_1X_2) = c_1(0) + c_1(0) = 0$$

and variance

$$\text{Var}(c_1X_1 + c_1X_2) = c_1{}^2(1) + c_1{}^2(1).$$

Hence for the variance to be 1, and the resulting distribution to be a standard normal, $c_1 = 1/\sqrt{2}$. Now consider $c_2\left({X_3}^2 + {X_4}^2 + {X_5}^2\right)^{1/2}$, we know that the sum of squares of three standard normals shall be χ^2 with three degrees of freedom which is the desired distribution of Y where $\nu = 3$, hence c_2 should be equal to $1/\sqrt{\nu}$ which is $1/\sqrt{3}$ in this case. Note that Z is a function of X_1, X_2, and Y is function of X_3, X_4, X_5, hence $Z \perp Y$. Thus

$$\frac{1/\sqrt{2}(X_1+X_2)}{1/\sqrt{3}\left({X_3}^2+{X_4}^2+{X_5}^2\right)^{1/2}} = \frac{Z}{\left(\frac{Y}{3}\right)^{1/2}} \sim t_3 \implies c = \frac{1/\sqrt{2}}{1/\sqrt{3}} = \sqrt{3/2}.$$

□

13. (p.446 #5) Suppose that $X_1, \ldots, X_n$ form a random sample from an exponential distribution with parameter β. Show that $(2/\beta)\sum_{i=1}^n X_i$ has a χ^2 distribution with $2n$ degrees of freedom.

Solution. Let $Y_i = (2/\beta)\ X_i$ and let M_{X_i}, M_{Y_i} denote the moment generating functions of X_i, Y_i respectfully. Then by the properties of moment generating functions,

$$M_{Y_i}(t) = M_{X_i}\left(\frac{2}{\beta}\ t\right) = \frac{1}{1-\beta\ \frac{2}{\beta}t} = \frac{1}{1-2t} \quad \text{for } t < \frac{1}{2}.$$

Since the X_i's are independent and identically distributed, it follows that the Y_i's are independent and identically distributed and thus the moment generating function of the sum, M_Y ($Y = \sum_{i=1}^n Y_i$), is

$$M_Y(t) = \prod_{i=1}^n M_{Y_i}(t) = \left(\frac{1}{1-2t}\right)^n \quad \text{for } t < \frac{1}{2}.$$

As $Y = (2/\beta)\sum_{i=1}^n X_i$, we see that the moment generating function of $(2/\beta)\sum_{i=1}^n X_i$ can be written as

$$\left(\frac{1}{1-2t}\right)^{2n/2} \quad \text{for } t < \frac{1}{2},$$

but this is the moment generating function of a χ^2 distribution with $2n$ degrees of freedom. Therefore $(2/\beta)\sum_{i=1}^n X_i \sim {\chi^2}_{2n}$, as was to be shown. □

14. (p.446 #8) Suppose that $X_1, \ldots, X_{n+1}$ form a random sample from a normal distribution, and let

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i \quad \text{and} \quad T_n = \left[\frac{1}{n}\sum_{i=1}^n \left(X_i - \overline{X}_n\right)^2\right]^{1/2}.$$

Determine the value of a constant k such that the random variable $k(X_{n+1} - \overline{X}_n)/T_n$ will have a t distribution.

Solution. Define random variables Y and Z as $Z = (X_{n+1} - \overline{X}_n)/\sigma$ and $Y = (n/\sigma^2){T_n}^2$. Since Z a linear combination of independently distributed normal random variables, it has a normal distribution. Further,

$$E(Z) = \frac{1}{\sigma}(E(X_n+1) - E(\overline{X}_n)) = \mu - \mu = 0$$

and variance

$$\text{var}(Z) = \frac{1}{\sigma^2}\left(\text{var}(X_{n+1}) + \text{var}(\overline{X}_n)\right) = \frac{1}{\sigma^2}(\sigma^2 + \frac{\sigma^2}{n}) = (1 + \frac{1}{n}) = \frac{n+1}{n}.$$

Then

$$Z \sim N(0, \frac{n+1}{n}) \qquad \text{and} \qquad Y = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 = \frac{S_n{}^2}{\sigma^2} \sim \chi^2{}_{n-1}.$$

As Y is independent of X_{n+1} and $\overline{X}_n$, it follows that Y and Z are independent. Now consider another random variable U, which is defined by the relation

$$U = \frac{Z/sqrt(n+1)/n}{\left(\frac{Y}{n-1}\right)^{1/2}}.$$

It follows from the definition of the t distribution that U has a t distribution with $n-1$ degrees of freedom. The expression for U can then be written in the following form:

$$U = \frac{(X_{n+1} - \overline{X}_n)/(\sigma\sqrt{(n+1)/n})}{\frac{T_n}{\sigma}\left(\frac{n}{n-1}\right)^{1/2}} \qquad = \frac{(X_{n+1} - \overline{X}_n)\left(\frac{n}{n+1}\right)^{1/2}}{T_n\left(\frac{n}{n-1}\right)^{1/2}}.$$

Therefore $k(X_{n+1} - \overline{X}_n)/T_n$ will have a t distribution when

$$k = \left(\frac{n}{n+1}\right)^{1/2}\left(\frac{n-1}{n}\right)^{1/2} = \left(\frac{n-1}{n+1}\right)^{1/2}.$$

□

15. (p.446 #9) Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with mean μ and variance σ^2, and Y is an independent random variable having a normal distribution with mean 0 and variance $4\sigma^2$. Determine a function of $X_1, \ldots, X_n$ and Y that does not involve μ or σ^2 but has a t distribution with $n-1$ degrees of freedom.

Solution. If we define random variables W and Z by the relations $Z = Y/2\sigma$ and $W = \sum_{i=1}^{n}(X_i - \overline{X}_n)^2/\sigma^2$ then

$$Z \sim N(0,1) \qquad \text{and} \qquad W = \frac{S_n{}^2}{\sigma^2} \sim \chi^2{}_{n-1}.$$

As Y is independent of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, it follows that Z and W are independent. Now consider another random variable U, which is defined by the relation

$$U = \frac{Z}{\left(\frac{W}{n-1}\right)^{1/2}}.$$

It follows from the definition of the t distribution that U has a t distribution with $n-1$ degrees of freedom. The expression for U can then be written in the following form:

$$U = \frac{Y/2\sigma}{\frac{1}{\sigma}\left(\frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}{n-1}\right)^{1/2}} \qquad = \frac{Y}{2\left(\frac{S_n{}^2}{n-1}\right)^{1/2}}.$$

Thus we have a function of $X_1, \ldots, X_n$ and Y that does not involve μ or σ^2 but has a t distribution with $n-1$ degrees of freedom, as was to be determined. □