THE HÖRMANDER COMPANION

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As the title suggests, these notes are designed as a companion to Hörmander's now-classic book *An Introduction to Complex Analysis in Several Variables* [Hör90]. This book has survived for many decades because it is concise and efficient. On the flip side, these characteristics can make it hard for the student to read. It assumes considerable background and tends to leave details to the reader.

Thus, in addition to being a graduate course in the analysis of functions of one and several complex variables, this is a course on learning how to learn mathematics. Sometimes it is tempting to abandon a hard book or paper because one doesn't have all the background. This book, for example, uses the language of differential forms and the wedge product on the second page. If you haven't met these ideas before and hear that they are taught in a course on manifolds, you might think you need to wait until after such a course to read this book. Similarly, this book uses theorems commonly first encountered in a course on measure and integration theory. You might think that therefore such a course is also a prerequisite.

My perspective is that if you wait to read a book or paper until you have mastered all the prerequisites, you won't progress. Instead, when you come up against material assumed as background with which you are unfamiliar, aim to develop enough of a working knowledge to be able to proceed with your main goal. As your interests develop, it will become clear which auxiliary areas require more attention on your part and which are peripheral.

These notes are designed to help you learn to do this. They include some expository material that may make reading Hörmander's exposition easier, they include exercises that allow you to practice using definitions and results, and they help you fill in some of the details left to the reader.

1. Analytic Functions of One Complex Variable

Let Ω be an open subset of $\mathbb{C}$ and consider $f : \Omega \to \mathbb{C}$. Write f in terms of its real and imaginary parts, i.e., if $z = x + iy$, $f(z) = u(x, y) + iv(x, y)$ for real-valued functions u and v of the two real variables x and y. Hörmander defines f to be **analytic** on Ω if $f \in C^1(\Omega)$ and $\frac{\partial f}{\partial \bar{z}} = 0$, where $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. It then follows that for f analytic, $df = \frac{\partial f}{\partial z} dz$ for $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$.

Exercise 1. *Using this definition of analyticity, determine the system of (real) partial differential equations satisfied by u and v.*

In most undergraduate texts on complex analysis, analyticity is defined as follows:

Definition 1. *$f : \Omega \to \mathbb{C}$ is* **analytic** *at $z_0 \in \Omega$ if*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1}$$

exists, in which case the value is denoted by $f'(z_0)$. f *is said to be analytic on the open set* Ω *if it is analytic at every point of* Ω.

Exercise 2. *Consider a function* $f = u + iv$ *for which the above limit exists for some* $z_0 = x_0 + iy_0$. *Show that at this point,* u *and* v *satisfy the same system of partial differential equations identified in the previous exercise. (Hint: Consider restricted approaches to* z_0*.) Then show that this definition of* f' *agrees with what Hörmander would write as* $\frac{\partial f}{\partial z}$.

The point of Hörmander's approach is that it reflects the intuition that analytic functions are those functions on $\mathbb{C}$ which are independent of $\bar{z}$. Such intuition is reinforced by the following elementary exercises:

Exercise 3. *Define* $e^z := e^x \cos y + ie^x \sin y$. *Show that this defines an analytic function on all of* $\mathbb{C}$. *(Such functions are called* **entire***.)*

Exercise 4. *The* **modulus** *of a complex number* $z = x + iy$ *is its usual Euclidean distance from the origin and is denoted by* $|z|$. *Show that* $|z|^2 = z\bar{z}$, *and determine all points of* $\mathbb{C}$ *at which* $f(z) = |z|^2$ *is analytic.*

Exercise 5. *Show that* $f(z) = z^n$ *is entire for* $n \in \mathbb{N}$. *This can be done in a number of ways. In particular, either definition of analyticity can be used. Try it both ways.*

2. Stokes'/Green's Theorem, the Cauchy Integral Formula, and Consequences

Most undergraduate books in complex analysis take a long time working up to the Cauchy Integral Formula, and then usually only give partial proofs for rather special domains. Few of these presentations do what Hörmander does and obtain the result from a version of Stokes'/Green's Theorem using the language of differential forms and the exterior derivative. Furthermore, these latter topics are often treated in a similarly incomplete manner in the undergraduate curriculum. Such pedagogical choices are appropriate and understandable; a thorough treatment of Stokes' theorem is usually left to a course in differential topology or smooth manifolds.

We will commit similar sins here and neither prove Stokes' theorem nor develop the underlying ideas fully. Such a development would be inappropriate here because what is really being used is the more elementary special case of Stokes' Theorem known as Green's Theorem. On the other hand, consistent with the spirit of this exposition, we aim to move the reader one step closer to an understanding of this deep result by introducing, albeit informally, notions of manifold, tangent space, and differential form, followed by two statements of Green's theorem and the proof of the Cauchy Integral Formula given in Hörmander's Section 1.2.

2.1. **What is a manifold?** Roughly speaking, a k-dimensional manifold is a set which near every point resembles $\mathbb{R}^k$. For example, consider the curve in the plane with equation $y = x^2$. This set is most certainly not a vector subspace of $\mathbb{R}^2$. However, this curve is locally linear; at every point along the curve there is a well-defined tangent line. We mention two other (trivial) classes of examples of manifolds: open subsets U of $\mathbb{R}^2$, and any finite set of points in $\mathbb{R}^2$.

For our purposes in this section, the three examples above are about all we need; we will consider curves in the plane which have a well-defined tangent line at all but finitely many points (e.g., the unit circle or the boundary of some polygonal region in the plane), open sets in the plane whose boundaries are these sorts of curves, and finite collections of points in the plane.

For any manifold, there is an associated notion of **dimension**, corresponding to the dimension of the Euclidean space it resembles locally. Thus an open set U in $\mathbb{R}^2$ is a manifold of dimension 2, the unit circle in $\mathbb{R}^2$ is a manifold of dimension 1, and a point or collection of points is a manifold of dimension 0. Also associated with any manifold is its **tangent space** at a point. In the case in which the manifold is also a subset of some $\mathbb{R}^n$, this notion of tangent space corresponds closely to the notion developed in calculus, except that because we want to think of the tangent space to a manifold at a point as a vector space, we always think of our tangent spaces as going through the origin.

Let us consider the three examples above. In all cases, since the manifold under consideration is a subset of $\mathbb{R}^2$, we will also think of the tangent space as a subspace of $\mathbb{R}^2$. If p is a point of an open subset U of $\mathbb{R}^2$, the tangent space to U at p, denoted $T_p(U)$, is just a copy of $\mathbb{R}^2$. For the point $p = (1/\sqrt{2}, 1/\sqrt{2})$ on the unit circle C, since the equation of the tangent line is $y - 1/\sqrt{2} = -(x - 1/\sqrt{2})$, the tangent space is given by $T_p(C) = \{\, (x, y) : y = -x \,\}$. Finally, the tangent space to the manifold consisting of the single point p is $T_p(\{p\}) = \{(0, 0)\}$.

Exercise 6. *Look up the definitions of a diffeomorphism, a smooth k-dimensional manifold $M \subset \mathbb{R}^n$, and the tangent space $T_p(M)$ to M at p. Using these definitions, show that the unit circle C in $\mathbb{R}^2$ is indeed a smooth 1-dimensional manifold and that (as claimed above) $T_{(1/\sqrt{2},1/\sqrt{2})}(C) = \{\, (x, y) : y = -x \,\}$. A good source is* [GP74]*, p.3 and p.9.*

2.2. **Tangent vectors and forms on $\mathbb{R}^n$.** Consider $\mathbb{R}^n$ as an n-dimensional manifold, or, more generally, consider an open subset U of $\mathbb{R}^n$. As discussed in the previous subsection, at any point p, the tangent space $T_p(\mathbb{R}^n)$ (resp. $T_p(U)$) is just $\mathbb{R}^n$ itself. We could denote the standard basis for this vector space by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, but to stress that *this* copy of $\mathbb{R}^n$ arises as the tangent space to a manifold at a point, a more common notation for this basis is

$$\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}.$$

Definition 2. *A* **1-form** *at $p \in \mathbb{R}^n$ (or U) is a linear map $\varphi : T_p(\mathbb{R}^n) \to \mathbb{R}$. The set of all 1-forms at p is the* **dual space** *to $T_p(\mathbb{R}^n)$ and is denoted $T_p^*(\mathbb{R}^n)$.*

The above is a special case of a standard construction in linear algebra:

Definition 3. *Let V be a normed vector space over $\mathbb{R}$. A* **linear functional** *is a linear map $\ell : V \to \mathbb{R}$. The set of all continuous linear functionals on V is called its* **dual space** *and is denoted by V^*.*

It is a standard result from linear algebra that the space dual to a vector space (equipped with the obvious operations) is itself a vector space, and when the original space has finite dimension n, so does the dual space. We denote by $\{dx_1, \ldots, dx_n\}$ the basis dual to the standard basis, so that the following relations hold:

$$dx_i(\frac{\partial}{\partial x_j}) = \delta_{i,j}.$$

More generally, we define k-forms on $\mathbb{R}^n$.

Definition 4. *A k-**form** on $\mathbb{R}^n$ (thought of as $T_p(\mathbb{R}^n)$ or $T_p(U)$) is a function $\varphi : (T_p(\mathbb{R}^n))^k \to \mathbb{R}$ that is multi-linear and anti-symmetric, i.e.,*

(a) *if $L^1, \ldots, L^k$ are vectors and $L^i = \alpha M^1 + \beta M^2$, then $\varphi(L^1, \ldots, \alpha M^1 + \beta M^2, \ldots, L^k) = \alpha\varphi(L^1, \ldots, M^1, \ldots, L^k) + \beta\varphi(L^1, \ldots, M^2, \ldots, L^k)$.*

(b) *$\varphi(L^1, \ldots, L^i, \ldots, L^j, \ldots, L^k) = -\varphi(L^1, \ldots, L^j, \ldots, L^i, \ldots, L^k)$.*

We denote the space of k-forms by $\Lambda^k(T_p^*(\mathbb{R}^n))$.

The first essential remark is that you are already quite familiar with one example of the above.

Example 5. *The determinant function* det *defines an n-form on $\mathbb{R}^n$. Indeed, to every collection of n vectors in $\mathbb{R}^n$, it assigns a real number. This function is linear in each entry, and if two vectors are interchanged, the determinant changes sign. In fact,* det *is the unique n-form on $\mathbb{R}^n$ for which* $\det(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) = 1$.

Not only does the determinant provide a familiar example of a form, but also in some sense all k-forms on $\mathbb{R}^n$ can be thought of as generalizations of the determinant.

We define an operation that allows us to combine forms to obtain a new form.

Definition 6. *Let φ be a k-form on $\mathbb{R}^n$ and η an l-form. We define $\varphi \wedge \eta : (T_p(\mathbb{R}^n))^{k+l} \to \mathbb{R}$ by*

(2)
$$\varphi\wedge\eta(L^1, \ldots, L^k, L^{k+1}, \ldots, L^{k+l}) = \sum_{\sigma \in S_{k+l}} sgn(\sigma)\varphi(L^{\sigma(1)}, \ldots, L^{\sigma(k)})\eta(L^{\sigma(k+1)}, \ldots, L^{\sigma(k+l)})$$

where the set S_{k+l} consists of all permutations of $\{1, \ldots, k + l\}$.

For example, if we consider the 1-forms dx_1 and dx_2 on $\mathbb{R}^2$, their wedge product $dx_1 \wedge dx_2$ is a 2-form on $\mathbb{R}^2$. If $L^1 = a\frac{\partial}{\partial x_1} + c\frac{\partial}{\partial x_2}$ and $L^2 = b\frac{\partial}{\partial x_1} + d\frac{\partial}{\partial x_2}$, then

$$dx_1 \wedge dx_2(L^1, L^2) = +dx_1(L^1)dx_2(L^2) - dx_1(L^2)dx_2(L^1) = ad - bc,$$

which we recognize as the determinant of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The following proposition contains properties of this wedge product.

Proposition 7. *Let φ, ω be k-forms, let η be an l-form, let τ be an r-form, and let α, β be scalars.*

(1) *$\varphi \wedge \eta$ is a $k + l$-form (i.e., it is multi-linear and antisymmetric).*
(2) *$(\alpha\varphi + \beta\omega) \wedge \eta = \alpha(\varphi \wedge \eta) + \beta(\omega \wedge \eta)$.*
(3) *$\varphi \wedge (\eta \wedge \tau) = (\varphi \wedge \eta) \wedge \tau$. We may thus use the notation $\varphi \wedge \eta \wedge \tau$ for either.*

Exercise 7. *Use the above definition to describe*

(1) *$dx_1 \wedge dx_1$*
(2) *the relationship between $dx_1 \wedge dx_2$ and $dx_2 \wedge dx_1$.*
(3) *$\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ where each φ_i is a 1-form on $\mathbb{R}^2$.*

State propositions generalizing the above for wedge products of forms on $\mathbb{R}^n$.

We make a few final comments in light of general results suggested by the above exercise. The only interesting spaces $\Lambda^k(T_p^*(\mathbb{R}^n))$ are for $k \leq n$, so henceforth when we talk about k-forms on $\mathbb{R}^n$, we will be thinking of $k \leq n$. Furthermore, there is an obvious vector space structure on $\Lambda^k(T_p^*(\mathbb{R}^n))$, and $\{\, dx^I := dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n \,\}$ is a basis.

2.3. **Differential forms and the exterior derivative.** For an open subset Ω of $\mathbb{R}^n$, let $\mathcal{E}(\Omega)$ be the space of infinitely-differentiable complex-valued functions on Ω.

Definition 8. *A* **differential** 1**-form** *or* **smooth** 1**-form** *on* Ω *is an object*

$$\alpha = \sum_{j=1}^{n} \alpha_j dx_j, \tag{3}$$

where $\alpha_j \in \mathcal{E}(\Omega)$.

One of the most familiar examples is the **exterior derivative** df of a function $f \in \mathcal{E}(\Omega)$.

Definition 9. *For* $f \in \mathcal{E}(\mathbb{R}^n)$, $p \in \mathbb{R}^n$, *and* $L \in T_p(\mathbb{R}^n)$, *define* df *by*

$$df(L) := L(f).$$

Of course it would be nice to know how to express df in the form (3). Taking L to be $\frac{\partial}{\partial x_j}$, one sees that $\alpha_j = \frac{\partial f}{\partial x_j}$, i.e.,

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j.$$

More generally,

Definition 10. *A* **smooth differential** k**-form on** Ω *(or, more succinctly, a* **smooth** k**-form***) is an object*

$$\varphi = \sum_{|I|=k} f^I dx^I \tag{4}$$

where each $f^I \in \mathcal{E}(\Omega)$, *each* I *is an increasing* k*-tuple of elements from the set* $\{ i : 1 \le i \le n \}$, *and the sum is taken over all increasing* k*-tuples. We denote by* $\mathcal{E}^k(\Omega)$ *the set of all smooth* k*-forms on* Ω. *Since smooth* 0*-forms are identified with smooth functions, the notations* $\mathcal{E}^0(\Omega)$ *and* $\mathcal{E}(\Omega)$ *may both be used for this space.*

Definition 11. *We define the* **exterior derivative** $d : \mathcal{E}^k(\Omega) \to \mathcal{E}^{k+1}(\Omega)$ *by*

$$d\left(\sum_{|I|=k} f^I dx^I\right) := \sum_{|I|=k} df^I \wedge dx^I.$$

Exercise 8.
(1) *Let* $\eta = x^2 y\, dx + xy^3\, dy$ *and* $\varphi = e^x(\cos y + i \sin y)\, dx - e^x(i \sin y + \cos y)\, dy$. *Compute* $d\eta$ *and* $d\varphi$.
(2) *Prove that for any smooth* k*-form* φ, $d^2\varphi = d(d\varphi) = 0$.
(3) *In a multivariable calculus course, one often learns about* **exact** *differential forms. Look up this definition and the connection between exact differential* 1*-forms and line integrals. Discuss how these ideas relate to the earlier pasts of this exercise.*
(4) *Look up the definition of a* **closed** *differential form, and discuss the connection between closed and exact forms.*

2.4. **The theorems.** The result you know as Green's Theorem is probably something close to the following:

Theorem 12 (Green's Theorem, First Version)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open, connected, and bounded set whose boundary $\partial\Omega$ is piece-wise smooth and positively-oriented (i.e., as one traverses the boundary in the positive sense, Ω lies to one's left). If P and Q are smooth throughout some open set containing $\overline{\Omega}$, then*

$$\int_{\partial\Omega} Pdx + Qdy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy.$$

This can be expressed in a different way using forms and exterior derivatives. For the 1-form $\varphi = Pdx + Qdy$ (using subscripts to denote partial derivatives)

$$\begin{aligned} d\varphi &= dP \wedge dx + dQ \wedge dy \\ &= (P_x dx + P_y dy) \wedge dx + (Q_x dx + Q_y dy) \wedge dy \\ &= P_x dx \wedge dx + P_y dy \wedge dx + Q_x dx \wedge dy + Q_y dy \wedge dy \\ &= -P_y dx \wedge dy + Q_x dx \wedge dy. \end{aligned}$$

We may thus restate Green's Theorem:

Theorem 13 (Green's Theorem, Second Version)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open, connected, and bounded set whose boundary $\partial\Omega$ is piece-wise smooth and positively-oriented. If φ is a smooth 1-form on a neighborhood of $\overline{\Omega}$,*

$$\int_{\partial\Omega} \varphi = \int_{\Omega} d\varphi.$$

Green's theorem is a special case of Stokes' theorem, which says more generally that for an oriented k-manifold M with boundary ∂M having the induced orientation, the integral of a $k-1$ form on the boundary equals the integral of its exterior derivative over the manifold. Stating it more precisely would require us to extend our definitions of vector, form, smooth differential form, and exterior derivative to functions defined on manifolds. This is "beyond the scope of this course."

We now return to the complex setting. Think of Ω as a connected, open, bounded subset of $\mathbb{C}$ with piece-wise smooth boundary. Consider $\varphi = fdz$ for $f \in C^1(\overline{\Omega})$. Then

$$\begin{aligned} d(fdz) &= df \wedge dz \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \wedge (dx + idy) \\ &= i\frac{\partial f}{\partial x} dx \wedge dy - \frac{\partial f}{\partial y} dx \wedge dy \\ &= 2i\frac{\partial f}{\partial \bar{z}} dx \wedge dy \\ &= \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz. \end{aligned}$$

Green's Theorem and this calculation thus give the following:

Corollary 14 (Cauchy Integral Theorem)**.** *If Ω and $\partial\Omega$ satisfy the same hypotheses as in Green's Theorem and if f is analytic in a neighborhood of $\overline{\Omega}$, then*

$$\int_{\partial\Omega} f(z)\, dz = 0.$$

Exercise 9. *Prove the identity* $2i dx \wedge dy = d\bar{z} \wedge dz$ *used above.*

Exercise 10. *Let* C *denote the positively-oriented unit circle. For each* $n \in \mathbb{Z}$, *evaluate (with justifications)* $\int_C z^n \, dz$. *Look up the definition of a* **simple closed curve** *in* $\mathbb{C}$. *What happens if* C *is replaced by a simple, closed, positively-oriented curve* γ *having the origin in its interior?*

We can now state the Cauchy Integral Formula.

Theorem 15 (Cauchy Integral Formula)**.** *If* $f \in C^1(\overline{\Omega})$,

$$2\pi i f(\zeta) = \int_{\partial\Omega} \frac{f(z)}{z-\zeta} dz + \int_{\Omega} \frac{\frac{\partial f}{\partial \bar{z}}}{z-\zeta} dz \wedge d\bar{z}. \tag{5}$$

Exercise 11. *Fill in the details of Hörmander's proof of this theorem. In particular, include complete justification of the statements*

$$\lim_{\varepsilon \to 0} \int_0^{2\pi} f(\zeta + \varepsilon e^{i\theta}) \, d\theta = 2\pi f(\zeta)$$

and

$$\lim_{\varepsilon \to 0} \iint_{\Omega_\varepsilon} \frac{\frac{\partial f}{\partial \bar{z}}}{z-\zeta} \, dz \wedge d\bar{z} = \iint_{\Omega} \frac{\frac{\partial f}{\partial \bar{z}}}{z-\zeta} \, dz \wedge d\bar{z}.$$

2.5. **Hörmander's Theorem 1.2.2.**

Theorem 16. *Let* μ *be a measure with compact support* K *in* $\mathbb{C}$. *The integral*

$$f(\zeta) = \int \frac{1}{z-\zeta} \, d\mu(z) \tag{6}$$

defines an analytic C^∞ *function on* $\mathbb{C} \setminus K$. *Furthermore, in any open set* Ω *where* $d\mu = (2\pi i)^{-1} \varphi dz \wedge d\bar{z}$ *for* $\varphi \in C^k(\Omega)$, *we have* $f \in C^k(\Omega)$ *with* $\frac{\partial f}{\partial \bar{z}} = \varphi$ *if* $k \geq 1$.

To summarize, this theorem tells us that (1) for ζ not in the support of μ, integrating against $1/(z-\zeta)$ produces an analytic function and (2) for ζ in the support of μ, if μ equals $(2\pi i)^{-1}\varphi dz \wedge d\bar{z}$, then f has the same degree of smoothness as φ and satisfies $\frac{\partial f}{\partial \bar{\zeta}} = \varphi$.

Understanding the theorem and its proof requires us to understand what a measure is and how to differentiate a function defined as an integral.

Measures. The theory of measure and integration is a course in itself. In this section, we give a definition of measure which should be enough to allow us to proceed.

Definition 17. *Let* X *be a compact subset of* $\mathbb{R}^n$ *and let* $C(X)$ *denote the vector space of complex-valued continuous functions on* X, *equipped with the sup-norm* $||f|| := \sup_{x \in X} |f(x)|$. *A* **complex measure on** X *is a continuous complex-valued linear functional on* $C(X)$.

If μ is a complex measure on X, we could denote its value on an $f \in C(X)$ by $\mu(f)$, but it is more traditional to denote it by $\int f(x) \, d\mu(x)$. The following exercise helps to explain why such notation is used.

Exercise 12. *Suppose* φ *is itself a continuous function on the compact subset* X *of* $\mathbb{R}^n$. *Show that*

$$f \mapsto \int f(x)\varphi(x) \, dx$$

defines a complex measure on X. *A common notation for this measure would be* $\varphi \, dx$.

In Theorem 1.2.2, Hörmander assumes that μ is *compactly supported*, meaning that there is a compact set K such that $\int f\,d\mu = \int_K f\,d\mu$. When $d\mu = \varphi\,dz \wedge d\bar{z}$ for φ continuous, this just means φ has compact support.

Differentiating functions defined by integrals. This is another common situation in analysis arising in this book for the first time in the proof of Theorem 1.2.2. We have a function defined by an integral:

$$f(\zeta) := \int \frac{1}{z-\zeta}\,d\mu(z), \tag{7}$$

where μ has compact support K. We are interested in whether f is differentiable or analytic as a function of ζ on $\mathbb{C} \setminus K$. Since for any ζ in $\mathbb{C} \setminus K$, $z - \zeta$ does not vanish for any z in K, viewed as a function of ζ, our integrand has the desired properties. Is the following legitimate?

$$\frac{\partial f}{\partial \bar{\zeta}} = \int \frac{\partial}{\partial \bar{\zeta}}\left(\frac{1}{z-\zeta}\right) d\mu(z) = \int 0\,d\mu(z) = 0$$

Fortunately, in many situations it is, though of course some justification is required. We discuss here several common arguments that may establish the legitimacy of differentiating under the integral sign.

The most obvious approach uses the definition of the derivative. For simplicity, suppose $f : \mathbb{R}^2 \to \mathbb{R}$. Define a new function $F : \mathbb{R} \to \mathbb{R}$ by $F(t) := \int f(x,t)\,dx$. If we wish to show that, under some hypotheses on f, $F'(t) = \int \frac{\partial f}{\partial t}(x,t)\,dx$, we must show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < |h| < \delta$,

$$\left| \frac{F(t+h) - F(t)}{h} - \int \frac{\partial f}{\partial t}(x,t)\,dx \right| < \varepsilon.$$

Furthermore, the left-hand-side of this inequality

$$\begin{aligned} &= \left| \int \frac{f(x,t+h) - f(x,t)}{h} - \frac{\partial f}{\partial t}(x,t)\,dx \right| \\ &= \left| \int \frac{1}{h} \left(\int_t^{t+h} \frac{\partial f}{\partial t}(x,s) - \frac{\partial f}{\partial t}(x,t)\,ds \right) dx \right|. \end{aligned}$$

Exercise 13. *State hypotheses on f under which the above argument can be completed to show that $F'(t) = \int \frac{\partial f}{\partial t}(x,t)\,dx$. Give the complete argument in this situation.*

Another approach makes use of limit theorems. After all, we are trying to determine whether

$$\begin{aligned} \lim_{h\to 0} \frac{F(t+h) - F(t)}{h} &= \lim_{h\to 0} \int \frac{f(x,t+h) - f(x,t)}{h}\,dx \\ &= \int \lim_{h\to 0} \frac{f(x,t+h) - f(x,t)}{h}\,dx. \end{aligned}$$

Exercise 14. *Many theorems exist giving hypotheses under which the limit of the integrals is the integral of the limit. The simplest such is usually encountered in an undergraduate analysis course. It says that the conclusion follows if $f_n \to f$* **uniformly**. *State the definition of uniform convergence for a sequence of functions on a set X, and then give a precise statement and proof of this theorem.*

Another more sophisticated limit theorem is the Dominated Convergence Theorem:

Theorem 18. *Let μ be a complex measure and let $\{f_n\}$ and f be integrable. Suppose for (almost) every x, $\lim_{n\to\infty} f_n(x) = f(x)$ and that there exists a function g with $\int |g|\, d\mu < \infty$ such that $|f_n(x)| \leq |g(x)|$ for all n and for (almost) every x. Then $\lim_{n\to\infty} \int f_n(x)\, d\mu = \int f\, d\mu$.*

In other words, this theorem says that the limit of a sequence of integrals is the integral of the limit if the integrands converge pointwise (almost everywhere) to the limit and if there is a single integrable function "dominating" all the functions in the sequence.

We use the Dominated Convergence Theorem to prove the following theorem on the continuity and differentiability of functions defined by integrals.

Theorem 19 ([Fol99], Theorem 2.27)**.** *Suppose $f : X \times (a,b) \to \mathbb{C}$ $(-\infty < a < b < \infty)$ and that $f(\cdot, t) : X \to \mathbb{C}$ is integrable for all $t \in (a,b)$. Consider $F(t) := \int_X f(x,t)\, d\mu(x)$.*

(a) *Suppose there exists $g : X \to \mathbb{C}$ such that $\int_X |g(x)|\, d\mu(x) < \infty$ and for all x, t, $|f(x,t)| \leq |g(x)|$. Then if $\lim_{t\to t_0} f(x,t) = f(x,t_0)$ for every x, $\lim_{t\to t_0} F(t) = F(t_0)$. In particular, continuity of $f(x,\cdot)$ for each x implies continuity of F.*

(b) *Suppose $\frac{\partial f}{\partial t}$ exists and there exists $g : X \to \mathbb{C}$ such that $\int_X |g(x)|\, d\mu(x) < \infty$ and for all x, t, $|\partial f / \partial t(x,t)| \leq |g(x)|$. Then F is differentiable and $F'(t) = \int \frac{\partial f}{\partial t}\, d\mu(x)$.*

Proof. For part (a), let $\{t_n\}$ be any sequence in $[a,b]$ such that $t_n \to t_0$. Set $g_n(x) := f(x,t_n)$. Then each g_n is integrable in x, $g_n(x) \to f(x,t_0)$ for every x, and $|g_n(x)| \leq |g(x)|$ for all x. Thus by the Dominated Convergence Theorem,

$$\begin{aligned}
\lim_{n\to\infty} F(t_n) &= \lim_{n\to\infty} \int_X f(x,t_n)\, d\mu(x) \\
&= \lim_{n\to\infty} \int_X g_n(x)\, d\mu(x) \\
&= \int_X f(x,t_0)\, d\mu(x) \\
&= F(t_0).
\end{aligned}$$

Next, consider part (b). Fix t. Let $\{h_n\}$ be a sequence of real numbers such that $h_n \to 0$. Set

$$g_n(x) := \frac{f(x,t+h_n) - f(x,t)}{h_n}.$$

Then each g_n is integrable in x, $\lim_{n\to\infty} g_n(x) = \frac{\partial f}{\partial t}(x,t)$ for all x, and $|g_n(x)| \leq |g(x)|$ for all n and x. Thus by the Dominated Convergence Theorem,

$$\begin{aligned}
\lim_{n\to\infty} \frac{F(t+h_n) - F(t)}{h_n} &= \lim_{n\to\infty} \int_X \frac{f(x,t+h_n) - f(x,t)}{h_n}\, d\mu(x) \\
&= \lim_{n\to\infty} \int_X g_n(x)\, d\mu(x) \\
&= \int_X \frac{\partial f}{\partial t}(x,t)\, d\mu(x).
\end{aligned}$$

Since this is true for all sequences $h_n \to 0$, $F'(t)$ exists and equals the integral of the derivative. □

Exercise 15. *Return to Hörmander's Theorem 1.2.2. If* $\zeta = t + is$*, show that* $\frac{\partial f}{\partial t}$ *and* $\frac{\partial f}{\partial s}$ *can be obtained by differentiating under the integral sign and that* $\frac{\partial f}{\partial \bar\zeta} = 0$ *on* $\mathbb{C}\setminus K$*. (If the presence of the measure makes you uncomfortable, assume throughout that* $d\mu = (2\pi i)^{-1}\varphi dz \wedge d\bar z$*.)*

Completing the Proof of Theorem 1.2.2. The above discussion and exercise establish the first statement of the Theorem, that $f(\zeta) = \int (z-\zeta)^{-1}\, d\mu(z)$ is C^∞ and analytic on $\mathbb{C} \setminus K$. We thus consider the second assertion.

Furthermore, we treat only the case $\Omega = \mathbb{R}^2$, so that for all z, $d\mu(z) = (2\pi i)^{-1}\varphi(z)dz\wedge d\bar z$ for φ in $C^k(\mathbb{R}^2)$ with compact support K. Then

$$\begin{aligned} f(\zeta) &= \frac{1}{2\pi i}\int \frac{\varphi(z)}{z-\zeta}\, dz \wedge d\bar z \\ &= \frac{1}{2\pi i}\int \frac{\varphi(z+\zeta)}{z}\, dz \wedge d\bar z. \end{aligned}$$

For every $\zeta = t+is$, the integrand is the product of a compactly-supported function and a locally-integrable function and is thus integrable. Since $\varphi \in C^k$ with compact support K, the integrand is a continuous function of t and s with the integrable upper bound $z^{-1}||\varphi||\chi_{K-\zeta}(z)$. By Theorem 19, f is continuous. If $k \geq 1$, since any derivative of φ up to order k is continuous and compactly supported, any derivative of the integrand in t and s up to order k is bounded by an integrable function of z alone. It follows from Theorem 19 again that $f \in C^k$ with derivatives of f obtained by differentiating under the integral sign.

Thus if $k \geq 1$,

$$\begin{aligned} \frac{\partial f}{\partial \bar\zeta}(\zeta) &= \frac{1}{2\pi i}\int \frac{\frac{\partial \varphi}{\partial \bar\zeta}(z+\zeta)}{z}\, dz \wedge d\bar z \\ &= \frac{1}{2\pi i}\int \frac{\frac{\partial \varphi}{\partial \bar z}(z+\zeta)}{z}\, dz \wedge d\bar z \\ &= \frac{1}{2\pi i}\int \frac{\frac{\partial \varphi}{\partial \bar z}(z)}{z-\zeta}\, dz \wedge d\bar z. \end{aligned}$$

The last integral is over all of $\mathbb{R}^2$, but since φ has compact support K, we obtain the same value if we integrate instead over an open disc $B(0,R)$ large enough that it contains K. Since φ is identically zero on $\partial B(0,R)$,

$$\begin{aligned} \frac{\partial f}{\partial \bar\zeta}(\zeta) &= \frac{1}{2\pi i}\int_{B(0,R)} \frac{\frac{\partial \varphi}{\partial \bar z}(z)}{z-\zeta}\, dz \wedge d\bar z \\ &= \frac{1}{2\pi i}\left(\int_{\partial B(0,R)} \frac{\varphi(z)}{z-\zeta}\, dz + \int_{B(0,R)} \frac{\frac{\partial \varphi}{\partial \bar z}(z)}{z-\zeta}\, dz \wedge d\bar z\right) \\ &= \varphi(\zeta), \end{aligned}$$

where for the last equality we have used the Cauchy Integral Formula. Theorem 1.2.2 is thus established when $\Omega = \mathbb{R}^2$.

2.6. **Consequences of the Cauchy Integral Formula.** It is not immediately obvious that the condition that an f in $C^1(\Omega)$ is also analytic is a particularly strong one. After all, we have seen that this is merely equivalent to complex differentiability of f, i.e., the existence of $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ for every $z_0 \in \Omega$. Just as real-valued functions can be once-differentiable but not twice-differentiable, it seems there must exist functions that are once complex-differentiable but not twice complex-differentiable.

This is, in fact, not the case. It turns out that the condition of complex differentiability is quite strong; analytic functions are remarkably well-behaved. As evidence, we establish a number of consequences of Hörmander's Theorems 1.2.1 and 1.2.2.

Corollary 20. *If* $f \in A(\Omega)$, $f \in C^\infty(\Omega)$ *and* $f' \in A(\Omega)$.

Proof. Fix $\zeta \in \Omega$. There exists $r > 0$ and $\zeta_0 \in \Omega$ such that $\zeta \in B(\zeta_0, r)$ and $\overline{B(\zeta_0, r)} \subseteq \Omega$. By the Cauchy Integral Formula,

$$\begin{aligned} f(\zeta) &= \frac{1}{2\pi i}\int_{\partial B(\zeta_0,r)} \frac{f(z)}{z-\zeta}\,dz \\ &= \int \frac{1}{z-\zeta}(2\pi i)^{-1} f(z)\chi_{\partial B(\zeta_0,r)}(z)dz\wedge d\bar{z}. \end{aligned}$$

Since $d\mu(z) := (2\pi i)^{-1} f(z)\chi_{\partial B(\zeta_0,r)}(z)dz \wedge d\bar{z}$ is a compactly-supported measure (supported on $\partial B(\zeta_0, r)$), by Theorem 1.2.2, f is C^∞ and analytic on $\mathbb{C}\setminus\partial B(\zeta_0, r)$, in particular at ζ. Consider $g := f'$. Then $g \in C^1(\overline{B(\zeta_0, r)})$ and

$$\frac{\partial g}{\partial \bar{\zeta}} = \frac{\partial^2 f}{\partial\bar{\zeta}\partial\zeta} = \frac{\partial}{\partial\zeta}\frac{\partial f}{\partial\bar{\zeta}} = 0$$

for all $\zeta \in B(\zeta_0, r)$. Thus f' is analytic at ζ and the result follows. □

Thus Hörmander uses his quite general Theorem 1.2.2 to conclude that an analytic f is in fact infinitely-differentiable and that all of its complex derivatives are analytic. One can obtain the result more directly:

Exercise 16 (Cauchy Integral Formula for Derivatives)**.** *Let f be analytic on Ω. For $\zeta \in \Omega$ and $\overline{B(\zeta, R)} \subseteq \Omega$, show without appealing to Theorem 1.2.2 that for any $j \in \mathbb{N}$,*

(8) $$f^{(j)}(\zeta) = \frac{j!}{2\pi i}\int_{\partial B(\zeta,R)} \frac{f(z)}{(z-\zeta)^{j+1}}\,dz.$$

The next exercise contains number of familiar and beautiful consequences of the above exercise. Of course any undergraduate text in complex analysis will have the proofs; try to prove them on your own.

Exercise 17. *Let $f : \Omega \to \mathbb{C}$ be analytic. Suppose $\zeta_0 \in \Omega$ and that $\overline{B(\zeta_0, R)} \subseteq \Omega$.*

(1) **(Cauchy Estimates)**

(9) $$|f^{(j)}(\zeta_0)| \le \frac{j!}{R^j}\sup_{|\zeta-\zeta_0|\le R}|f(\zeta)|.$$

(2) **(Liouville's Theorem)** *If f is bounded and entire, f is constant.*
(3) **(Fundamental Theorem of Algebra)** *If f is a non-constant polynomial over $\mathbb{C}$, there exists $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.*

It is also natural to ask for what notion of limit is the limit of a sequence of analytic functions analytic? We will prove the following:

Proposition 21. *If $f_n \in A(\Omega)$ and $f_n \to f$ uniformly on compact subsets of Ω, then $f \in A(\Omega)$.*

Exercise 18. *You may wish to consult an undergraduate analysis text such as* [Str00].

(1) *Let* $\{f_n\}$ *be a sequence of continuous functions from* $[a, b]$ *to* $\mathbb{R}$. *For what mode of convergence is the limit function* f *guaranteed to be continuous? State such a theorem and give an example showing that if the hypothesis is violated, the conclusion may fail.*
(2) *Let* $\{f_n\}$ *be a sequence of differentiable functions from* $[a, b]$ *to* $\mathbb{R}$. *Suppose* $f_n(x) \to f(x)$ *for all* x. *Under what additional hypotheses is* f *differentiable with* $f'(x) = \lim_{n\to\infty} f_n'(x)$*? State such a theorem and give an example showing that if the hypothesis is violated, the conclusion may fail.*

The proof of the proposition requires the next theorem on the size of derivatives of an analytic function.

Theorem 22 (Hörmander's Theorem 1.2.4)**.** *For every compact subset* K *of* Ω *and every open neighborhood* ω *of* K *in* Ω, *there exist constants* C_j *such that for all* $f \in A(\Omega)$,

$$\sup_{z\in K} |f^{(j)}(z)| \le C_j \int_\omega |f(z)|\, dx\, dy. \tag{10}$$

Proof. Fix a compact subset K and a bounded open neighborhood ω of K with $\omega \subseteq \Omega$.

Let $C_0^\infty(\omega)$ denote the set of all infinitely-differentiable functions with compact support contained in ω. Let $\psi \in C_0^\infty(\omega)$ with the additional property that $\psi \equiv 1$ on a neighborhood ω' of K. Consider $f \in A(\Omega)$ and define $g := f\psi$. We want to apply the Cauchy Integral Formula to g on ω. Since $g(z) \equiv 0$ on $\partial\omega$ and $\frac{\partial g}{\partial \bar z} = f \frac{\partial \psi}{\partial \bar z}$,

$$\begin{aligned} g(\zeta) &= f(\zeta)\psi(\zeta) \\ &= \frac{1}{2\pi i} \int_\omega \frac{f(z)\frac{\partial\psi}{\partial\bar z}(z)}{z-\zeta}\, dz \wedge d\bar z. \end{aligned}$$

Since ψ is a smooth function with compact support, its first partial derivatives are bounded functions. Thus there exists M satisfying $|\partial\psi/\partial\bar z| \le M$ on ω. Furthermore, since $\partial\psi/\partial\bar z \equiv 0$ on ω', for $\zeta \in K$ there exists $d > 0$ such that $|z - \zeta| \ge d$ on the support of the integrand. Thus for $\zeta \in K$

$$\begin{aligned} |f(\zeta)| &= |g(\zeta)| \\ &= \int_\omega \left| f(z) \frac{\frac{\partial\psi}{\partial\bar z}(z)}{z-\zeta} \right| dxdy \\ &\le \frac{M}{d} \int_\omega |f(z)|\, dxdy. \end{aligned}$$

The result follows for $j = 0$. □

Exercise 19. *Finish the proof by first establishing that for* $j \in \mathbb{N}$ *and* $\zeta \in K$,

$$f^{(j)}(\zeta) = \frac{a_j}{2\pi i} \int_\omega \frac{f(z)\frac{\partial\psi}{\partial\bar z}(z)}{(z-\zeta)^{j+1}}\, dz \wedge d\bar z.$$

We now prove Proposition 21.

Proof. Fix a compact set K in Ω and an open neighborhood ω of K with compact closure in Ω. (This is expressed succinctly as $K \subset \omega \subset\subset \Omega$.) Applying the previous theorem to $f_n - f_m$ gives

$$\sup_{z\in K} |f'_n(z) - f'_m(z)| \leq C_1 \int_\omega |f_n(z) - f_m(z)|\, dxdy.$$

Since the sequence $\{f_n\}$ is uniformly convergent, it is uniformly Cauchy, and hence by the Dominated Convergence Theorem, the right-hand side of the above inequality tends to zero. The sequence $\{\partial f_n/\partial z\}$ is thus uniformly convergent on K.

Since the f_n are analytic,

$$\frac{\partial f_n}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f_n}{\partial x} + i\frac{\partial f_n}{\partial y}\right) = 0,$$

i.e., $\frac{\partial f_n}{\partial x} = -i\frac{\partial f_n}{\partial y}$. Thus $\frac{\partial f_n}{\partial z} = i\frac{\partial f_n}{\partial y} = \frac{\partial f_n}{\partial x}$ and the uniform convergence of $\{\partial f_n/\partial z\}$ on K implies the uniform convergence of the sequences $\{\partial f_n/\partial x\}$ and $\{\partial f_n/\partial y\}$. We now invoke the real-variable result to conclude that $\frac{\partial f}{\partial x} = \lim_{n\to\infty} \frac{\partial f_n}{\partial x}$ and $\frac{\partial f}{\partial y} = \lim_{n\to\infty} \frac{\partial f_n}{\partial y}$, so that $\frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}) = \lim_{n\to\infty} \frac{1}{2}(\frac{\partial f_n}{\partial x} + i\frac{\partial f_n}{\partial y}) = 0$. □

As a consequence, we obtain

Corollary 23. *Suppose the power series $f(z) = \sum_{j=0}^\infty a_j z^j$ converges on $B(0,R)$. Then $f \in A(B(0,R))$.*

Proof. We will show that for every $0 < r < R$, the series $\sum_{j=0}^\infty a_j z^j$ converges uniformly and absolutely on $\overline{B(0,r)}$. Recall that to say that this series **converges uniformly** on $\overline{B(0,r)}$ means that the sequence $\{f_n\}$ of partial sums, defined by $f_n(z) := \sum_{j=0}^n a_j z^j$, converges uniformly to f on $\overline{B(0,r)}$. Uniform convergence on every closed ball $\overline{B(0,r)}$ in $B(0,R)$ would then imply that $f_n \to f$ uniformly on compact subsets of $B(0,R)$. Since the f_n are polynomials in z, they are analytic, and so Proposition 21 would give the analyticity of the limit function f.

The proof that a power series whose convergence set includes $B(0,R)$ converges uniformly and absolutely on every $\overline{B(0,r)} \subset B(0,R)$ is left to the reader as an exercise. □

Exercise 20. *Go to it, reader! You may begin by recalling or proving that if we have a series $\sum u_j$ of functions all defined on a common ball B, and if $\sum M_j$ is a convergent series of non-negative numbers, then if $|u_j| \leq M_j$ throughout B, the series $\sum u_j$ converges uniformly and absolutely on B. This is sometimes called the* **Weierstrass M-test**.

We are now ready to prove the converse of the above, which is yet another result illustrating our point in this section that the condition of analyticity is remarkably strong.

Theorem 24 (Hörmander's Theorem 1.2.8)**.** *If $f \in A(B(0,R))$, we have*

$$f(z) = \sum_{j=0}^\infty \frac{f^{(j)}(0)}{j!} z^j, \tag{11}$$

with uniform convergence on every compact subset of $B(0,R)$.

Proof. For $0 < r < \rho < R$, by the Cauchy Integral Formula (with a change in notation), for $|z| \leq r$,

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Observe,

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta} \frac{1}{(1 - \zeta^{-1} z)} \\ &= \frac{1}{\zeta} \sum_{j=0}^{\infty} \zeta^{-j} z^j, \end{aligned}$$

where we have used the formula for the sum of a convergent geometric series.

Define $g_n(z,\zeta) := \frac{f(\zeta)}{\zeta} \sum_{j=0}^{n} \zeta^{-j} z^j$ and $g(z,\zeta) := \frac{f(\zeta)}{\zeta - z}$. Since

$$\begin{aligned} |g_n(z,\zeta)| &\leq \frac{\sup_{|\zeta|=\rho} |f(\zeta)|}{\rho} \sum_{j=0}^{n} \rho^{-j} r^j \\ &= \frac{\sup_{|\zeta|=\rho} |f(\zeta)|}{\rho} \frac{1}{1 - \frac{r}{\rho}}, \end{aligned}$$

$g_n \to g$ uniformly. The following is therefore legitimate:

$$\begin{aligned} \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \frac{1}{2\pi i} \int_{|\zeta|=\rho} \lim_{n\to\infty} g_n(z,\zeta)\, d\zeta \\ &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_{|\zeta|=\rho} g_n(z,\zeta)\, d\zeta \\ &= \lim_{n\to\infty} \sum_{j=0}^{n} \left(\frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta^{j+1}}\, d\zeta \right) z^j \\ &= \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j. \end{aligned}$$

□

For functions that are merely in $C^\infty(\Omega)$, knowing the function in a neighborhood of a single point tells us nothing about its behavior in other parts of the domain. The situation is radically different for those functions represented by power series.

Corollary 25 (Uniqueness of analytic continuation)**.** *As usual, let $\Omega \subset \mathbb{C}$ be a connected open set. If $f \in A(\Omega)$ and if there exists $z_0 \in \Omega$ such that $f^{(j)}(z_0) = 0$ for all $j \geq 0$, then $f \equiv 0$ on Ω.*

Proof. Let $Z := \{\, z \in \Omega : f^{(j)}(z) = 0, j = 0, 1, 2, \dots \}$. Since $Z = \bigcap_{j=0}^{\infty} (f^{(j)})^{-1}(\{0\})$ and each function $f^{(j)}$ is continuous, Z is a closed subset of Ω.

On the other hand, by hypothesis, Z is not empty since $z_0 \in Z$. Fix $\zeta \in Z$ and consider g defined by $g(z) := f(z + \zeta)$. g is analytic in some neighborhood $B(0,R)$ of the origin, and on this disc it is represented by the series $\sum f^{(j)}(z_0) z^j / j!$. Thus $g \equiv 0$ on $B(0,R)$, i.e., $f \equiv 0$ on $B(\zeta, R)$. This proves that Z is open in Ω. Since Z is both open and closed in Ω and Ω is connected, $Z = \Omega$. □

Exercise 21 (Laurent series)**.** *Let $0 \le r < R$ and define the annulus $a(r,R) := \{ z : r < |z| < R \}$. If f is analytic on this annulus, then*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \qquad a_j = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta^{j+1}} \, d\zeta,$$

where $r < \rho < R$. (Suggestion: Apply the Cauchy Integral Formula to $a(r', R')$ for $r < r' < R' < R$ and use the idea of the proof of Theorem 24.)

2.7. **Mean-Value and Maximum Modulus Properties.** Whereas the consequences of the Cauchy Integral formula discussed in the previous subsection largely concern regularity properties of analytic functions, the results of this section begin to explore the extent to which, for $f \in A(\overline{\Omega})$, we can relate its behavior on Ω to its values on $\partial\Omega$.

To begin with, if $f \in A(\Omega)$ and $\overline{B(\zeta, R)} \subset \Omega$, then

$$\begin{aligned} f(\zeta) &= \frac{1}{2\pi i} \int_{\partial B(\zeta,R)} \frac{f(z)}{z-\zeta} \, dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\zeta + Re^{i\theta})}{Re^{i\theta}} iRe^{i\theta} \, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\zeta + Re^{i\theta}) \, d\theta. \end{aligned}$$

This is the **mean value property** for analytic functions: for a function analytic on $\overline{B(\zeta, R)}$, its value at the center of the disc is equal to the average of its values on the boundary. From this, the maximum modulus principle will follow.

Theorem 26 (Maximum Principle - local version)**.** *Suppose f is analytic on $B(\zeta, R)$. If for all $z \in B(\zeta, R)$, $|f(z)| \le |f(\zeta)|$, then f is constant on $B(\zeta, R)$.*

Proof. If $f(\zeta) = 0$, then $|f(z)| \le |f(\zeta)|$ on $B(\zeta, R)$ gives $f \equiv 0$ on $B(\zeta, R)$. Thus in the remainder of the proof we may suppose $f(\zeta) \ne 0$. Fix $0 < r < R$. By the mean value property,

$$f(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta + re^{i\theta}) \, d\theta.$$

Equivalently,

$$0 = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \, d\zeta - \frac{1}{2\pi} f(\zeta + re^{i\theta}) \, d\theta,$$

or, dividing by $f(\zeta)$ and writing the right-hand side as a single integral,

$$0 = \frac{1}{2\pi} \int_0^{2\pi} 1 - \frac{f(\zeta + re^{i\theta})}{f(\zeta)} \, d\theta.$$

It follows that

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left(1 - \frac{f(\zeta + re^{i\theta})}{f(\zeta)} \right) d\theta. \tag{12}$$

Write $z = \zeta + re^{i\theta}$ and observe

$$\begin{aligned}
2\,\mathrm{Re}\left(1 - \frac{f(z)}{f(\zeta)}\right) &= 1 - \frac{f(z)}{f(\zeta)} + 1 - \overline{\frac{f(z)}{f(\zeta)}} \\
&= 2 - \frac{f(z)\overline{f(\zeta)} - \overline{f(z)}f(\zeta)}{|f(\zeta)|^2} \\
&= \frac{2|f(\zeta)|^2 - f(z)\overline{f(\zeta)} - \overline{f(z)}f(\zeta)}{|f(\zeta)|^2} \\
&\geq \frac{|f(\zeta)|^2 - f(z)\overline{f(\zeta)} - \overline{f(z)}f(\zeta) + |f(z)|^2}{|f(\zeta)|^2} \\
&= \frac{|f(\zeta) - f(z)|^2}{|f(\zeta)|^2} \\
&\geq 0.
\end{aligned}$$

Return to equation (12). The integrand is a *continuous* function of θ. Furthermore, the last calculation shows it to be non-negative. Thus it can only have integral zero if it is zero for all θ, i.e., $f(\zeta + re^{i\theta}) = f(\zeta)$ for all θ. Since $r < R$ was arbitrary, the result follows. □

Theorem 27 (Maximum Principle). *Suppose f is analytic on a domain $\Omega \subset\subset \mathbb{C}$ and f extends continuously to $\partial\Omega$. Then f attains its maximum modulus on $\partial\Omega$.*

Exercise 22. *Prove* **Schwarz's Lemma***: Let f be analytic on $B(0,1)$ with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in B(0,1)$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in B(0,1)$. If $|f'(0)| = 1$ or if there exists $z_0 \in B(0,1)$ such that $|f(z_0)| = |z_0|$, then there exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $f(z) = \alpha z$ for all $z \in B(0,1)$.*

3. Runge's Theorem

This theorem is often skipped in a first course in complex variables. If you would like to read a more elementary and more extensive discussion of the theorem than that given by Hörmander, see Conway's book [Con78]. In fact, we will follow Conway to some degree here.

Look again at Theorem 24. One way to restate it is to say that every analytic function on $B(0,R)$ is the uniform limit on compact subsets of a sequence of polynomials. We consider generalizations. For instance, if we replace $B(0,R)$ by an arbitrary open set G, is it true that for every $f \in A(G)$ and every compact $K \subset G$, f can be uniformly approximated on K by polynomials? As the next example shows, the answer is no.

Example 28. *Take $G := \{\, z : 0 < |z| < 2 \,\}$. Then $f(z) = z^{-1}$ is analytic on G. Suppose we had a sequence $\{p_n\}$ of polynomials such that $p_n \to f$ uniformly on compact subsets of G. In particular, since $K := \partial B(0,1)$ is a compact subset of G,*

$p_n \to f$ uniformly on $\partial B(0,1)$. Thus (if the circle is oriented counter-clockwise)

$$\begin{aligned} 2\pi i &= \int_{\partial B(0,1)} z^{-1}\,dz \\ &= \int_{\partial B(0,1)} \lim_{n\to\infty} p_n(z)\,dz \\ &= \lim_{n\to\infty} \int_{\partial B(0,1)} p_n(z)\,dz \\ &= 0. \end{aligned}$$

Since this is a contradiction, no such sequence of polynomials exists.

We make two remarks concerning this example. First, neither G^c nor K^c is connected. Second, in the notation of Exercise 21, $G = a(0,2)$, and so every analytic function on G is a limit (uniform on compact subsets) of a sequence of rational functions with poles outside of G.

In light of the above, we might try to characterize those G with the property that every analytic function on G can be approximated uniformly on compact subsets by polynomials. We might also look for a more general theorem on the approximation of analytic functions on a set G by rational functions with poles outside G.

Theorems of this sort vary in their precise formulation but tend to go by the name of Runge's Theorem. In many texts such as Conway's [Con78] Runge's Theorem is very clearly about the uniform approximation of analytic functions on a compact set by polynomials, whereas in Hörmander's text, no explicit mention is made of rational functions. We include both statements here so that the reader may compare them. We will only prove the version stated by Hörmander.

Definition 29. *Consider $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. We put a topology on this set by taking as a basis for the topology all open discs $B(a,r)$ in $\mathbb{C}$ together with sets of the form $\{z : |z| > r\} \cup \{\infty\}$. Because $\mathbb{C}_\infty$ with this topology is homeomorphic to the 2-sphere S^2, we visualize $\mathbb{C}_\infty$ this way and call it the* **Riemann sphere**.

Theorem 30 (Runge's Theorem [Con78])**.** *Let K be a compact subset of $\mathbb{C}$ and let E be an open subset of $\mathbb{C}_\infty \setminus K$ that meets each component of $\mathbb{C}_\infty \setminus K$. Let Ω be an open set containing K. If $f \in A(\Omega)$ and if $\varepsilon > 0$ is given, then there exists a rational function R, all of whose poles lie in E, such that $\sup_{z\in K} |f(z) - R(z)| < \varepsilon$.*

Theorem 31 (Runge's Theorem [Hör90])**.** *Let Ω be an open set in $\mathbb{C}$ and let K be a compact subset of Ω. The following conditions on Ω and K are equivalent:*

(a) *Every function analytic in a neighborhood of K can be approximated uniformly on K by functions in $A(\Omega)$.*
(b) *The open set $\Omega \setminus K$ has no component which is relatively compact in Ω.*
(c) *For every $\zeta \in \Omega \setminus K$ there exists $f \in A(\Omega)$ such that*

$$|f(\zeta)| > \sup_K |f|. \tag{13}$$

If we take $\Omega = \mathbb{C}$, we obtain a theorem on the uniform approximation of analytic functions by polynomials.

Corollary 32. *Let K be a compact subset of $\mathbb{C}$. Every function analytic in a neighborhood G of K can be uniformly approximated by polynomials on K if and only if K^c is connected, or, equivalently, if and only if for every $\zeta \in K^c$ there exists a polynomial p in z such that $|p(\zeta)| > \sup_K |p|$.*

Upon a first reading, you may find the statement of this theorem somewhat intimidating. This is a normal response. It is also likely that jumping right into the proof will not help. Something useful to do at this point is to look at some examples. Examples don't prove theorems, but they are the key way we build intuition.

For simplicity, let us take Ω to be all of $\mathbb{C}$. Define two compact subsets: $K_1 := \overline{a(1/2, 3/2)}$ and $K_2 := \overline{B(0,1)}$.

Consider first $\Omega \setminus K_1$. This has two components, namely $B(0,1/2)$ and $\{z : |z| > 3/2\}$. The first component, call it O, has closure $\{z : |z| \leq 1/2\}$, which is a compact subset of Ω. Therefor the pair (Ω, K_1) violates (b). If the theorem is true, we should be able to see that (a) and (c) are violated as well. Indeed, the example preceding the statements of the theorems essentially addresses (a); the function $f(z) = z^{-1}$ is analytic on $\mathbb{C} \setminus \{0\}$, which is a neighborhood of K_1, but this function is not the uniform limit on K_1 of a sequence of entire functions.

To see that (c) is also violated, we must find $\zeta \in \Omega \setminus K_1$ such that for all $f \in A(\Omega)$, $|f(\zeta)| \leq \sup_{K_1} |f|$. Take $\zeta = 0$ and any entire f (since $\Omega = \mathbb{C}$). Then since f is analytic on any $\overline{B(0,r)}$, by the maximum principle,

$$|f(0)| \leq \sup_{|z|=r} |f(z)|,$$

and in particular

$$|f(0)| \leq \sup_{1/2 \leq |z| \leq 3/2} |f(z)| = \sup_{K_1} |f|.$$

On the other hand, $\Omega \setminus K_2 = \{z : |z| > 1\}$. This set is connected and its closure is $\{z : |z| \geq 1\}$, which is certainly not compact in Ω. Thus (b) is satisfied, and if the theorem is true, (a) and (c) must hold as well. Concerning (a), we have already proved that if G is an open set containing $\overline{B(0,1)}$, any function analytic on G is the uniform limit on the compact subset $\overline{B(0,1)}$ of the sequence of polynomials consisting of the partial sums of the power series representing the analytic function.

To see that (c) holds, note that if $\zeta \in \Omega \setminus K_2$, then $|\zeta| > 1$. Consider $f(z) = z$. This is entire and

$$|\zeta| = |f(\zeta)| > 1 = \sup_{K_2} |f|.$$

With these examples in mind, we turn to the proof of the theorem.

Exercise 23. *Hörmander begins by proving that (c) implies (b). Read this part of his proof, and write out a version of it for yourself in which you fill in any details you need. Focus especially on the statement that, if O is a component of $\Omega \setminus K$ with compact closure in Ω, then for every $f \in A(\Omega)$,*

$$\sup_{\overline{O}} |f| \leq \sup_{K} |f|. \tag{14}$$

Proof of $(a) \Rightarrow (b)$. This will be a proof by contradiction. Suppose (b) fails, and let O denote a component of $\Omega \setminus K$ with compact closure in Ω. Since (a) is valid, for any f analytic in a neighborhood of K, there exists a sequence $\{f_n\}$ of elements of $A(\Omega)$ such that $f_n \to f$ uniformly on K. Inequality (14) yields

$$\sup_{\overline{O}} |f_n - f_m| \leq \sup_{K} |f_n - f_m|.$$

Thus $\{f_n\}$ is also uniformly convergent on $\overline{O}$. Let F denote the limit function. Since F is the uniform limit on compact subsets of O of a sequence of analytic functions, F is analytic on O. As a uniform limit of continuous functions, it is also continuous on ∂O. Furthermore, since F and f are both continuous on ∂O and are the limit of the same sequence there, $F = f$ on ∂O.

The above argument applies to any f analytic in a neighborhood of K. Thus if $\zeta \in O$, $f(z) := \frac{1}{z-\zeta}$ is analytic in the neighborhood $\mathbb{C} \setminus \{\zeta\}$ of K, and we obtain an associated F analytic in O and continuous on ∂O with $F(z) = \frac{1}{z-\zeta}$ on ∂O. Set $g(z) := (z - \zeta)F(z)$. This function is analytic in O and is identically 1 on the boundary. Thus $g \equiv 1$ on O. Since $\zeta \in O$ and $g(\zeta) = 0$, this is a contradiction, proving that (a) implies (b).

Exercise 24. *Prove the assertion made in the penultimate sentence of the preceding paragraph: If O is a bounded open set and if g is analytic on O and continuous on ∂O, then if $g \equiv c$ on ∂O, then $g \equiv c$ on O.*

Proof of *(b)* $\Rightarrow$ *(a)*. Hörmander intends to apply the Hahn-Banach Theorem in this part of the proof. See Rudin's book [Rud66] for a statement of this theorem as well as its application in the proof of Runge's Theorem. We follow that development here. Recall,

Theorem 33 (Hahn-Banach, [Rud66])**.** *If M is a subspace of a normed vector space X and if λ is a bounded linear functional on M, then λ can be extended to a bounded linear functional Λ on all of X without increasing the norm, i.e., $||\Lambda|| = ||\lambda||$.*

The following is a consequence:

Proposition 34. *Let M be a subspace of a normed vector space X and let $x_0 \in X$. Then x_0 is in the closure $\overline{M}$ of M if and only if there exists no bounded linear functional Λ on X with $\Lambda(x) = 0$ for all x in M but $\Lambda(x_0) \neq 0$.*

Exercise 25. *Prove Proposition 34.*

We are now prepared to understand Hörmander's application of the Hahn-Banach theorem. For K and Ω as in part (b), consider $C(K)$, the vector space of continuous functions on K equipped with the sup-norm. Thus $C(K)$ plays the role of the normed vector space X above. Let M be the subspace consisting of the restrictions of elements of $A(\Omega)$. Let f be analytic on a neighborhood of K. Then $f \in X$, and the question of whether or not it is the uniform limit on K of elements of $A(\Omega)$ is equivalent to the question of whether or not it is in $\overline{M}$. By Proposition 34, to prove that $f \in \overline{M}$, it suffices to prove that for every continuous linear functional μ on X with $\mu(g) = 0$ for every $g \in A(\Omega)$, $\mu(f) = 0$ as well. Since the set of all continuous linear functionals on K is precisely the set of complex measures, we must show that if μ is a measure such that

$$\int g \, d\mu = 0 \quad \text{for all } g \in A(\Omega),$$

then $\int f \, d\mu = 0$.

Exercise 26. *With this background, work through the rest of Hörmander's proof of the Runge approximation theorem. Write a summary of the key steps of the arguments that (b) implies (a) and that (b) implies (c).*

We restate Runge's Theorem to motivate the next definition: Given a domain Ω, some compact subsets have the right topological properties that guarantee that analytic functions on them are close to analytic functions on Ω whereas other compact subsets do not have this property. Given a compact subset K of Ω without this property, it would be nice to be able to enlarge it to satisfy the hypotheses of Runge's Theorem. Moreover, condition (c) in Runge's Theorem gives us another way to describe those points we must add to K. We thus make the following definition.

Definition 35. *Let K be a compact subset of Ω. We define $\widehat{K}$, the $A(\Omega)$-**hull** of K by*

$$\widehat{K} = \widehat{K}_\Omega := \{\, z \in \Omega : |f(z)| \leq \sup_K |f| \quad \text{for every } f \in A(\Omega)\,\}. \tag{15}$$

This set has the following properties:

Proposition 36. *Let K be a compact subset of Ω, $\widehat{K}$its $A(\Omega)$-hull.*

(a) $K \subseteq \widehat{K}$.
(b) *$\widehat{K}$ satisfies the hypotheses of Runge's theorem.*
(c) $d(K, \Omega^c) = d(\widehat{K}, \Omega^c)$.
(d) *If $ch(K)$ denotes the convex hull of K, $\widehat{K} \subseteq ch(K)$.*

Exercise 27. *Prove part (c) of Proposition 36. One direction is immediate. For the other, follow Hörmander's suggestion: For $\zeta \in \Omega^c$, consider the function $f(z) = (z-\zeta)^{-1}$.*

Proof of (b). This simply requires us to unwind the definitions. To show that $\widehat{K}$ satisfies the hypotheses of Runge's Theorem, we must show that for all $\zeta \in \Omega \setminus \widehat{K}$, there exists $f \in A(\Omega)$ with $|f(\zeta)| > \sup_K |f|$. If this were not the case, we could find $\zeta_0 \in \Omega \setminus \widehat{K}$ such that for all $f \in A(\Omega)$, $|f(\zeta_0)| \leq \sup_K |f|$. But then such a ζ_0 would be an element of $\widehat{K}$. This contradiction establishes the claim.
Proof of (d). We discuss part (d) because it presents us with an opportunity to begin our exploration of notions related to convexity. We recall a number of definitions.

Definition 37. *A subset E of $\mathbb{R}^n$ is **convex** if for all $x, y \in E$, the line segment joining x and y is contained in E, i.e., $(1-t)x + ty \in E$ for all $t \in [0,1]$.*

Definition 38. *If $E \subseteq \mathbb{R}^n$, the **convex hull of** E, denoted $ch(E)$, is the intersection of all convex sets containing E.*

We can describe the convex hull of a set E in another way. We focus here on E closed since this is the situation we have in mind, though one can formulate similar results for more general E. A **hyperplane** in $\mathbb{R}^n$ is given by $\{\, x : \sum a_j x_j = c\,\}$, $a \in \mathbb{R}^n$. Such a hyperplane divides $\mathbb{R}^n$ into two closed half-spaces $\overline{H^+} := \{\sum a_j x_j \geq c\}$ and $\overline{H^-} := \{\sum a_j x_j \leq c\}$ and is said to be a support hyperplane for a set E if E is contained in one of the half-spaces and if there exists $y \in E$ such that $\sum a_j y_j = c$.

Exercise 28. *Let $E \subseteq \mathbb{R}^n$ be closed.*

(1) *If y is exterior to $ch(E)$, then there exists a hyperplane $\sum a_j x_j = c$ such that $E \subset \overline{H^-}$ and $\sum a_j y_j > c$. Such a hyperplane* **separates** *E and $\{y\}$.*
(2) *If $ch(E)$ is not all of $\mathbb{R}^n$, then $ch(E)$ is the intersection of all support closed half-spaces containing it.*

With this background, we are ready to prove part (d) of Proposition 36. Take $z_0 = x_0 + iy_0 \in \widehat{K}$ and suppose that z_0 is not in $\mathrm{ch}(K)$. Then there exists $(a,b) \in \mathbb{R}^2$ and $c \in \mathbb{R}$ such that $ax + by \leq c$ for all $z = x + iy \in K$ but $ax_0 + by_0 > c$. Since $\mathrm{Re}[(a - ib)(x + iy)] = ax + by$, if we set $\alpha = a + ib$, we have

$$\mathrm{Re}(\bar{\alpha} z) \leq c \quad \text{but} \quad \mathrm{Re}(\bar{\alpha} z_0) > c.$$

Now, for any complex α, $z \mapsto e^{\bar{\alpha} z}$ is entire, hence in $A(\Omega)$ for any Ω. Furthermore, for all $z \in K$

$$\begin{aligned} |e^{\bar{\alpha} z}| &= e^{\mathrm{Re}(\bar{\alpha} z)} \\ &\leq e^c \\ &< e^{\mathrm{Re}(\bar{\alpha} z_0)} \\ &= |e^{\bar{\alpha} z_0}|. \end{aligned}$$

This contradicts the assumption that $z_0 \in \widehat{K}$, hence $\widehat{K} \subseteq \mathrm{ch}(K)$. This completes the proof of part (d) of the proposition.

The topological description of $\widehat{K}$. We close this section with a different description of the set $\widehat{K}$ which matches with the intuition that $\widehat{K}$ "fills in the holes" of K.

Theorem 39 (Hörmander, Theorem 1.3.3)**.** *$\widehat{K}_\Omega$ is the union of K with the components of $\Omega \setminus K$ that are relatively compact in Ω.*

Proof. Suppose that O is a component of $\Omega \setminus K$ that is relatively compact in Ω. Then by inequality (14), $O \subset \widehat{K}$. Thus if K_1 is the union of K with such relatively compact subsets of $\Omega \setminus K$, then $K_1 \subseteq \widehat{K}$. We wish to establish the reverse containment.

Since K_1 is the union of K with components of $\Omega \setminus K$, $\Omega \setminus K_1$ is open. It follows that K_1 is closed and in fact compact. Furthermore, $\Omega \setminus K_1$ has no components which are relatively compact in Ω. Thus K_1 satisfies (b) in Runge's theorem, and hence (c). Thus we conclude $K_1 = \widehat{K_1}$. Since $\widehat{K} \subseteq \widehat{K_1}$, we conclude $\widehat{K} \subseteq K_1$, as desired. □

3.1. **Meromorphic Functions and the Mittag-Leffler Theorem.** This is another context in which Hörmander's treatment differs substantially from what is typically done in a course on functions of one complex variable. Thus in this section we begin by giving the more elementary definitions.

Here, we follow Section 1.4.2 of [Var11]. We begin by defining three kinds of singularities for functions analytic on a punctured disc.

Definition 40. *Let f be analytic on a punctured disc $B'(\zeta, r) := \{ z : 0 < |z - \zeta| < r \}$ about ζ, with Laurent series $\sum_{j=-\infty}^{\infty} a_j (z - \zeta)^j$ about ζ.*

1. *f has a* **pole of order** *$n \geq 1$ if $a_{-n} \neq 0$ but $a_{-k} = 0$ for all $k \geq n$.*
2. *f has a* **removable singularity** *at ζ if $a_j = 0$ for all $j < 0$.*
3. *f has an* **essential singularity** *at ζ if there are infinitely many $j < 0$ for which $a_j \neq 0$.*

Exercise 29. *Find all poles of $f(z) = \frac{1}{z(z^2+1)^2}$ and determine the order of each. Then determine $\int_\gamma f(z)\, dz$ where γ is a positively-oriented circle of radius $R > 1$. (You probably know a result called the Residue Theorem that allows you to do this quickly. If you want to use that theorem, prove it first. If you don't recall* *that theorem, approach this problem by considering the integral over a contour that includes the circle of radius R together with circles of radius ε about the poles.)*

Exercise 30. *Suppose f is analytic on $B'(\zeta, r)$.*

(1) *If f is bounded in a neighborhood of ζ, show that the Laurent series for f about ζ has no non-zero coefficients a_j for $j < 0$.*
(2) *Prove that f has a pole at ζ if and only if $\lim_{z \to \zeta} |f(z)| = \infty$.*

This last exercise shows that, for a function with a pole at ζ, it makes sense to assign the value ∞ at ζ, so that such a function could be viewed as a function into $\mathbb{C} \cup \{\infty\}$. The same can not be said of a function with an essential singularity at ζ.

With this background, we have the following "standard" definition of a meromorphic function.

Definition 41. *Let $\Omega \subseteq \mathbb{C}$. f is* **meromorphic at** *$\zeta \in \Omega$ if there is a punctured disc centered at ζ on which f is analytic and if ζ is not an essential singularity of f. f is* **meromorphic on** *Ω if it is meromorphic at every point of Ω.*

Since an analytic function h can only vanish to at most finite order at a point, it is clear that the ratio of analytic functions defined on a neighborhood of ζ is meromorphic at ζ. We will eventually address the question of whether every meromorphic functions is a ratio of analytic functions.

Now that we have reviewed the classical treatment of meromorphic functions, we discuss Hörmander's formulation of the concept.

Fix $z \in \mathbb{C}$. We consider the set of all functions which are analytic on some neighborhood of z. We define an equivalence relation on this set so that $f \sim g$ if there is a neighborhood of z on which f is identically equal to g. We let A_z be the set of equivalence classes under this relation, and denote the equivalence class determined by f by f_z. With addition and multiplication defined in the obvious way, A_z is a ring. Furthermore, it has no zero divisors, and we may form the quotient field $M_z := \{ f_z/g_z : f_z, g_z \in A_z, g_z \neq 0_z \}$.

Exercise 31. *Let A_z be as defined above.*

(1) *How should we define $f_z + g_z$? Show that (according to your definition) $f_z + g_z$ is in fact well-defined.*
(2) *Give all the details of the argument that A_z has no zero divisors.*

We can now state Hörmander's definition of a meromorphic function.

Definition 42. *Let Ω be an open subset of $\mathbb{C}$. A* **meromorphic function** *on Ω is a mapping $\varphi : \Omega \to \cup_z M_z$ such that for all z in Ω, $\varphi(z) \in M_z$ and such that for all points z_0 in Ω there is a neighborhood ω and elements $f, g \in A(\omega)$ such that $\varphi(z) = f_z/g_z$ for all $z \in \omega$. We denote the set of all meromorphic functions on Ω by $M(\Omega)$ and write φ_z instead of $\varphi(z)$ for $\varphi \in M(\Omega)$.*

The aspect of this definition that matches our intuition for meromorphic functions is its explicit reference to quotients of (equivalence classes of) analytic functions. On the other hand, its formulation in terms of equivalence classes may make you wonder what connection these objects have with analytic functions.

The first essential remark, then, is that $A(\Omega)$ can be identified with a subset of $M(\Omega)$ in a natural way. Take $F \in A(\Omega)$. For any $z \in \Omega$, since Ω is open, F is indeed analytic in a neighborhood of z and hence determines an equivalence class F_z. Since the ring A_z is naturally embedded in M_z, F_z is identified with an element of M_z. Consider the map φ defined by $z \mapsto F_z$. In order for this to be a meromorphic function, there is another condition it must satisfy. Let z_0 be a point of Ω and consider ω a neighborhood of z_0 in Ω. Since $F \in A(\omega)$ as well, it is indeed the case that there is a single pair of analytic functions F and 1 on ω for which $\varphi_z := F_z/1_z$ for all $z \in \omega$. We conclude that the φ defined in this way is indeed a meromorphic function.

In order for $A(\Omega)$ to be identified with a subset of $M(\Omega)$, we need to also know that distinct elements of $A(\Omega)$ are identified with different elements of $M(\Omega)$. Thus suppose $F, G \in A(\Omega)$ and that these are not identically equal. Then there exists $z_0 \in \Omega$ such that $F(z_0) \neq G(z_0)$. With respect to the equivalence relation associated with functions analytic in a neighborhood of z_0, these two functions are not equivalent, and so they determine different equivalence classes. That is, $F_{z_0} \neq G_{z_0}$ in A_{z_0}, and so $F_{z_0}/1_{z_0} \neq G_{z_0}/1_{z_0}$ in M_{z_0}.

As a next step in reconciling Hörmander's definition of meromorphic function with the classical one, we ask whether it is possible to sensibly assign an element of $\mathbb{C} \cup \{\infty\}$ to $q \in M_\zeta$. If we can do so, this will allow us to identify a meromorphic function on Ω with a map from Ω to $\mathbb{C} \cup \{\infty\}$.

Thus take $q \in M_\zeta$, so that there exist $f_\zeta, g_\zeta \in A_\zeta$ with $g_\zeta \neq 0_\zeta$ such that $q = f_\zeta/g_\zeta$. If $f_\zeta = 0_\zeta$, we define $q(\zeta) := 0$. Suppose, then, that $f_\zeta \neq 0_\zeta$. Let f and g be representatives of f_ζ and g_ζ. Both are analytic in a neighborhood of ζ, and neither is identically zero, so there exist integers $n, m \geq 0$ such that

$$f(z) = (z-\zeta)^n f_1(z) \qquad g(z) = (z-\zeta)^m g_1(z),$$

where f_1 and g_1 are analytic in a neighborhood of ζ with $f_1(\zeta) \neq 0$ and $g_1(\zeta) \neq 0$. We define

$$q(\zeta) := \begin{cases} 0 & n-m>0 \\ f_1(\zeta)/g_1(\zeta) & m-n=0 \\ \infty & n-m<0. \end{cases} \tag{16}$$

We must show that $q(\zeta)$ is in fact well-defined. Thus suppose there are different analytic functions F and G on a neighborhood of ζ so that $q = F_\zeta/G_\zeta$, $G_\zeta \neq 0_\zeta$. We must show that this leads to the same value for $q(\zeta)$. If $f_\zeta/g_\zeta = F_\zeta/G_\zeta$, then also

$$f_\zeta G_\zeta = F_\zeta g_\zeta,$$

and this holds in A_ζ. Since neither g_ζ nor G_ζ is 0_ζ, $f_\zeta = 0_\zeta$ if and only if $F_\zeta = 0_\zeta$.

It suffices, then, to consider the case in which none of the four functions is identically zero in a neighborhood of ζ. There exist non-negative integers n,N, m, and M and analytic functions f_1, g_1, F_1 and G_1 not vanishing at ζ so that $f(z) = (z-\zeta)^n f_1(z)$, etc. Therefore

$$(z-\zeta)^n f_1(z)(z-\zeta)^M G_1(z) = (z-\zeta)^N F_1(z)(z-\zeta)^m g_1(z).$$

Since each side represents an analytic function in a neighborhood of ζ and since equality holds throughout this neighborhood, by the uniqueness theorem for analytic functions, $n+M = N+m$, i.e., $n-m = N-M$. It is now clear that the definition of $q(\zeta)$ is independent of the particular choices made in representing q.

ADD STUFF ON ANALYTICITY OF MEROMORPHIC FUNCTIONS.

The above discussion can be summarized by saying that, if F is meromorphic in an open set containing ζ, on a sufficiently small neighborhood of ζ, F is a quotient of analytic functions. This is somewhat consistent with our intuition, but is a local result.

Still, we have an immediate corollary on the Laurent expansion for a function meromorphic at ζ:

Theorem 43.

The expression $\sum_{k=1}^{n} A_k(z-\zeta)^{-k}$ is the **principal part** of F at ζ. Our next theorem, known as the Mittag-Leffler Theorem, tells us that we can find a single meromorphic function on Ω with prescribed principal parts at each point of a discrete subset of Ω.

Theorem 44 (Mittag-Leffler). *Let $D := \{ z_j : j \in \mathcal{J} \}$ ($\mathcal{J}$ a finite or countable index set) be a discrete subset of the open subset Ω of $\mathbb{C}$, and for each j, let f_j be meromorphic in a neighborhood of z_j. Then there exists $f \in M(\Omega)$ such that f is analytic on $\Omega \setminus D$ and for all j, $f - f_j$ is analytic in a neighborhood of z_j.*

The proof applies the Runge approximation theorem to each member of an increasing sequence of compact subsets of Ω. We thus first state the lemma that establishes the existence of such a sequence.

Lemma 45. *Let Ω be an open (not necessarily connected) subset of $\mathbb{C}$. Then there exists a collection $\{K_n\}$ of compact sets such that $\Omega = \cup_n K_n$ and*

(1)
(2)
(3) .

We leave the details of the proof to the reader, but give an outline of the argument. For each n, set

$$V_n := B(\infty, n) \cup \bigcup_{a \in (\mathbb{C} \setminus \Omega)} B(a, 1/n).$$

Set $K_n := S^2 \setminus V_n$.

3.2. **Harmonic Functions.** We recall that the **Laplacian** on $\mathbb{R}^n$ is the differential operator

$$\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

Definition 46. *Let Ω be an open subset of $\mathbb{R}^n$. Then $h \in C^2(\Omega)$ is* **harmonic** *if $\Delta h \equiv 0$ on Ω.*

Exercise 32. *Show that when $n = 2$, $\Delta = 4\partial^2/\partial z \partial \bar{z}$.*

Note that if f is analytic on Ω, it follows that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are harmonic. It is also true that every harmonic function u is the real part of some analytic function f. (See the next exercise for some further discussion.) Hence the mean value property for analytic functions implies a corresponding mean value property for harmonic functions:

Proposition 47. *If $\Omega \subseteq \mathbb{R}^2$ and if h is harmonic on Ω, then if $\overline{B(z,r)} \subseteq \Omega$,*

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(z + re^{i\theta}) \, d\theta. \tag{17}$$

Exercise 33 (harmonic conjugates)**.** *If u is a harmonic function on $\mathbb{C}$, a* **harmonic conjugate** *is a function v for which $f = u + iv$ is analytic. One method for finding a harmonic conjugate of a harmonic function u uses the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. (e.g., Example 1.5.20 in* [MH99]*.) A second method observes that we seek f with $f(z) + \overline{f(z)} = 2u(x, y)$. If u is real analytic, it is legitimate to replace x with $\frac{z+\bar{z}}{2}$, y with $\frac{z-\bar{z}}{2i}$, and then to treat z and $\bar{z}$ as independent and set $\bar{z} = 0$. This gives $f(z) + \overline{f(0)} = 2u(\frac{z}{2}, \frac{z}{2i})$. (See p.93 of* [D'A10]*.) Use each of these methods to obtain the harmonic conjugates of each of the following:*

(1) $u(x, y) = x^3 - 3xy^2$.
(2) $u(x, y) = e^x \cos(y)$.

3.3. **Subharmonic Functions.** Section 2.1 of Krantz's book [Kra92] includes a rather similar treatment of subharmonic functions. One difference is that we tend to focus on subharmonic functions on $\mathbb{R}^2$ or $\mathbb{C}$ whereas Krantz proves his theorems in $\mathbb{R}^N$.

Many definitions of subharmonic functions exist, and different definitions are possible depending on the degree of smoothness one assumes. We will consider functions which are, to begin with, upper-semicontinuous.

Definition 48. *Let $\Omega \subset \mathbb{R}^n$ be open. $u : \Omega \to [-\infty, \infty)$ is* **upper-semicontinuous** *if for every $s \in \mathbb{R}$, $\{ x \in \Omega : u(x) < s \}$ is open.*

Exercise 34.
If $E \subset \mathbb{R}^n$, we define its **characteristic function**, *χ_E, by the condition $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ otherwise. State a hypothesis on E under which χ_E is usc. g is a step function on $[a, b] \subseteq \mathbb{R}$ if there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$ and real numbers c_j, $1 \leq j \leq n$, such that $g(x) = c_j$ on (x_{j-1}, x_j). How must g be defined at x_j in order for g to be usc?*

The following proposition includes several results about upper-semicontinuous functions that Hörmander uses but does not state explicitly.

Proposition 49. *Suppose $u : \Omega \subseteq \mathbb{R}^n \to [-\infty, \infty)$.*

(1) *u is usc on Ω if and only if for every x_0, $\limsup_{x \to x_0} u(x) \leq u_(x_0)$.*
(2) *If u is usc, u is the point-wise limit of a decreasing sequence of continuous functions.*
(3) *If u is usc and K is compact, $\sup_K u$ is attained at some point of K.*

Definition 50. *$u : \Omega \to [-\infty, \infty)$ is* **subharmonic** *if (i) u is upper-semicontinuous and (ii) for every compact $K \subset \Omega$, whenever h is continuous on K and harmonic on Int(K) with $u \leq h$ on ∂K, $u \leq h$ on K.*

As a little exercise, think about what this means for functions on $\mathbb{R}$. Harmonic functions are those with second derivative zero, i.e., linear functions. Then a subharmonic function on $\mathbb{R}$ is one with the property that, if its graph lies on or below a line on the boundary of a compact set K, the same is true throughout K. Thus subharmonicity is can be thought of as a generalization of convexity.

We next consider Hörmander's Theorem 1.6.3, giving a number of necessary and sufficient conditions for subharmonicity.

Theorem 51. *Let $u : \Omega \subseteq \mathbb{C} \to [-\infty, \infty)$ be upper-semicontinuous. The following are necessary and sufficient for u to be subharmonic.*

(a) *If D is a closed disc contained in Ω and if f is an analytic polynomial with $u \le \mathrm{Re}(f)$ on ∂D, then $u \le \mathrm{Re}(f)$ on D.*

(b) *Let $\Omega_\delta := \{\, z \in \Omega : d(z, \Omega^c) > \delta \,\}$. Then if $z \in \Omega_\delta$,*

$$u(z) \le \frac{1}{2\pi} \frac{\int \int_0^{2\pi} u(z + re^{i\theta})\, d\theta\, d\mu(r)}{\int d\mu(r)}, \tag{18}$$

where μ is any positive measure supported in $[0, \delta]$.

(c) *For every $\delta > 0$ and every $z \in \Omega_\delta$, there exists some positive measure μ supported in $[0, \delta]$ with some mass away from the origin for which* (18) *is valid.*

We begin with a few remarks. Recall that if h is harmonic on Ω, it satisfies the mean value property, i.e., if $\overline{B(z, \delta)} \subset \Omega$ and $0 < r \le \delta$,

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(z + re^{i\theta})\, d\theta.$$

We could, if we like, integrate both sides against $d\mu(r)$ if μ is a measure on $[0, \delta]$. A natural choice would be to take $d\mu(r) = \chi_{[0,\delta]}(r) r\, dr$. This gives

$$h(z) \int_0^\delta r\, dr = \frac{1}{2\pi} \int_0^\delta \int_0^{2\pi} h(z + re^{i\theta}) r\, d\theta\, dr.$$

The integral on the left gives the value $\frac{1}{2}\delta^2$. On the right, we obtain a double integral, in polar coordinates, over the disc $B(z, \delta)$. Simplifying yields

$$h(z) = \frac{1}{\pi\delta^2} \int_{w \in B(z,\delta)} h(w)\, dA.$$

The right-hand side is the average value of h over the disc $B(z, \delta)$. Thus we have a obtained a second "mean value property" from the first through integration, and, furthermore, it is apparent other equalities may be established by integrating with respect to different measures. Part (b) of the above theorem is the analogous inequality for subharmonic function.

Exercise 35. *Read Hörmander's proof of Theorem 1.6.3. State all lemmas pertaining to trigonometric polynomials and upper-semicontinuous functions used in this proof. See also Proposition 49 above. Select two of these lemmas and give their proofs.*

We close this section with a discussion of the connection between subharmonicity of a function f and and the non-negativity of the Δf. This part of the exposition follows Krantz closely, though we treat the special case of Ω open in $\mathbb{C}$ rather than in $\mathbb{R}^N$ for an arbitrary natural number N.

We will use without proof the following lemma:

Lemma 52. *If f is subharmonic on Ω and not identically $-\infty$, then f is locally integrable on Ω and $\mathcal{P} := \{\, z \in \Omega : f(z) = -\infty \,\}$ has Lebesgue measure zero.*

Proposition 53. *Suppose f is subharmonic and not identically $-\infty$. Suppose $\varphi \in C_0^\infty(\Omega)$ is non-negative. Then*

$$\int f \Delta\varphi\, dA \ge 0.$$

Proof. Take $z \in \Omega$ and $r > 0$ such that $\overline{B(z,r)} \subseteq \Omega$. By the sub-mean value property for subharmonic functions (established as part of the proof of Hörmander's Theorem 1.6.3),

$$f(z) \leq \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta.$$

Multiply by $\varphi(z)$ (which is assumed non-negative) and integrate over $\mathbb{C}$. The local integrability of f on Ω and the compact support of φ in Ω guarantee that the integrals are absolutely convergent. We obtain

$$\int f(z)\varphi(z)\,dA \leq \frac{1}{2\pi}\int\int_0^{2\pi} f(z+re^{i\theta})\varphi(z)\,d\theta\,dA.$$

We change variables in the area integral, setting $w = z + re^{i\theta}$ and then expand φ about w in a Taylor polynomial of degree 2 plus an error E_2 with the property that $\lim_{r\to 0+} r^{-2}E_2(r) = 0$. We obtain the following string of inequalities:

$$\begin{aligned}
&\int f\varphi\,dA \\
\leq\;& \frac{1}{2\pi}\int f(w)\int_0^{2\pi}\varphi(w-re^{i\theta})\,d\theta\,dA \\
=\;& \frac{1}{2\pi}\int f(w)\int_0^{2\pi}\varphi(w) - \frac{\partial\varphi}{\partial x}(w)r\cos\theta - \frac{\partial\varphi}{\partial y}(w)r\sin\theta + \frac{1}{2}\frac{\partial^2\varphi}{\partial x^2}(w)r^2\cos^2\theta \\
&\quad + \frac{\partial^2\varphi}{\partial x\partial y}(w)r^2\cos\theta\sin\theta + \frac{1}{2}\frac{\partial^2\varphi}{\partial y^2}(w)r^2\sin^2\theta + E_2(r)\,d\theta\,dA \\
=\;& \frac{1}{2\pi}\int f(w)\left[2\pi\varphi(w) + \frac{\pi}{2}\frac{\partial^2\varphi}{\partial x^2}(w)r^2 + \frac{\pi}{2}\frac{\partial^2\varphi}{\partial y^2}(w)r^2 + \int_0^{2\pi}E_2(r)\,d\theta\right]dA.
\end{aligned}$$

Thus

$$0 \leq \int f(w)\left[\frac{1}{4}\Delta\varphi(w)r^2 + \frac{1}{2\pi}\int_0^{2\pi}E_2(r)\,d\theta\right]dA$$

Dividing by r^2 and taking the limit as $r \to 0^+$ gives the result. □

Corollary 54. *If $f \in C^2(\Omega)$ and f is subharmonic, $\Delta f \geq 0$.*

Exercise 36. *Prove Corollary 54. Use the preceding proposition; for φ fixed as in the proposition, its support is contained in some rectangle $[a,b] \times [c,d] \subseteq \mathbb{R}^2$. Integrate by parts in the integral $\int f\Delta\varphi\,dA$.*

4. Elementary Properties of Functions of Several Complex Variables

4.1. **Definitions and notation.** The development in this section is analogous to the one-variable case. We denote the coordinates on $\mathbb{C}^n$ by $z = (z_1,\ldots,z_n)$ and write either $z_j = x_{2j-1} + ix_{2j}$ or $z_j = x_j + iy_j$, $1 \leq j \leq n$. Then

$$\begin{aligned}
dz_j &= dx_{2j-1} + idx_{2j} \\
d\bar{z}_j &= dx_{2j-1} - idx_{2j}.
\end{aligned}$$

Then if $\Omega \subseteq \mathbb{C}^n$ is open and $u \in C^1(\Omega)$, we may write

$$du = \sum_{k=1}^{2n}\frac{\partial u}{\partial x_k}dx_k$$

or we may write du in terms of the $2n$ independent 1-forms dz_j and $d\bar{z}_j$ as

$$du = \sum_{j=1}^{n} \frac{\partial u}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j. \tag{19}$$

Inspired by this decomposition, we make a number of definitions.

Definition 55. *Suppose φ is a 1-form. φ is* **of type** $(1,0)$ *if it is a combination of the dz_j alone and is* **of type** $(0,1)$ *if it is a combination of the $d\bar{z}_j$ alone. The $(1,0)$ part of du in (19) is denoted by ∂u whereas the $(0,1)$ part of du is denoted by $\bar{\partial} u$.*

Definition 56. *$u \in C^1(\Omega)$ is* **analytic** *(synonymously,* **holomorphic***) if du is of type $(1,0)$, i.e., if $\bar{\partial} u = 0$.*

More generally, we define the type or bi-degree of higher-degree differential forms.

Definition 57. *A differential form f on $\Omega \subset \mathbb{C}^n$ is* **of type** (p,q) *if*

$$f = \sum_{|I|=p} \sum_{|J|=q} f_{IJ} dz^I \wedge d\bar{z}^J,$$

where the f_{IJ} are smooth functions on Ω, $I = (i_1, \ldots, i_p)$ for $1 \le i_1 < \ldots < i_p \le n$ is an increasing p-tuple and $dz^I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$. J and $d\bar{z}^J$ are interpreted similarly.

As in one-variable case, we may take the exterior derivative of the form f:

$$\begin{aligned} df &:= \sum_{|I|=p} \sum_{|J|=q} df_{IJ} \wedge dz^I \wedge d\bar{z}^J \\ &= \sum_{|I|=p} \sum_{|J|=q} (\partial f_{IJ} + \bar{\partial} f_{IJ}) \wedge dz^I \wedge d\bar{z}^J \\ &= \sum_{|I|=p} \sum_{|J|=q} \partial f_{IJ} \wedge dz^I \wedge d\bar{z}^J + \sum_{|I|=p} \sum_{|J|=q} \bar{\partial} f_{IJ} \wedge dz^I \wedge d\bar{z}^J. \end{aligned}$$

We denote the first sum on the last line by ∂f and the second by $\bar{\partial} f$. Note that if f is of type (p,q), ∂f is of type $(p+1,q)$ and $\bar{\partial} f$ is of type $(p,q+1)$.

Exercise 37. *Previously we have shown that $d^2 = 0$. Use this to show that $\partial^2 = 0$, $\bar{\partial}\partial + \partial\bar{\partial} = 0$, and $\bar{\partial}^2 = 0$*

We end this section on definitions and notation by discussing the topology in $\mathbb{C}^n$ in more detail. As expected, we equip $\mathbb{C}^n$ with the hermitian inner product $\langle z, w \rangle := \sum_{j=1}^{n} z_j \bar{w}_j$, which induces a norm and metric space structure. One therefore defines the open ball $B(z_0, r) := \{ z : ||z - z_0|| < r \}$ and naturally thinks of such sets as a basis for the open sets. When $n = 1$, this definition even reduces to that of an open disc in $\mathbb{C}$. There is, however, another basis of open sets generalizing open discs - the polydiscs. For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $r = (r_1, \ldots, r_n)$, $r_j > 0$, define

$$D(a,r) := \{ z : |z_j - a_j| < r_j \} = D(a_1, r_1) \times \cdots \times D(a_n, r_n).$$

Observe that we now have two different notations for a disc in $\mathbb{C}$. Note also that in the notation $B(a,r)$, r is the radius and is a single number, whereas in the notation $D(a,r)$, if r is an n-tuple of non-negative real numbers. The topological boundary of $D(a, r)$ is $\{\, z : |z_j - a_j| = r_j \quad \text{for } some\ j \,\}$, whereas the **distinguished boundary** $\partial_0 D(a, r)$ is $\{\, z : |z_j - a_j| = r_j \quad \text{for } all\ j \,\}$.

Although the collection of open balls and the collection of open polydiscs are a basis for the same topology, it is worth noting even here that a ball and a polydisc have different function theory. It is an important and non-obvious result in several complex variables that the ball and the polydisc are not biholomorphically equivalent.

4.2. **The Cauchy Integral Formula and its Consequences.** In section 2.2, Hörmander turns his attention to the several-variable version of the Cauchy Integral Formula and its consequences. His discussion is somewhat complicated by the fact that he also wishes to explore the connection between separate analyticity and analyticity.

Recall the definition of an analytic function on $\Omega \subset \mathbb{C}^n$. We begin by considering $f \in C^{(}\Omega)$ and say that f is analytic if $\bar{\partial} f = 0$. This means $\partial f / \partial \bar{z}_j = 0$ for all $1 \leq j \leq n$. In particular, f is what we might call **separately analytic** - when $n - 1$ of the variables are held fixed and f is considered as a function of the single remaining complex variable as it ranges over some slice of Ω, the resulting function of one complex variable is once continuously differentiable and satisfies the Cauchy-Riemann equations in that variable.

We might ask about the converse. If we f is defined on $\Omega \subseteq \mathbb{C}^n$ is separately analytic, is it in fact analytic on Ω? This appears to be a weaker hypothesis because we do not begin with a function which is $C^1(\Omega)$ - it is only C^1 as a function of each variable individually when the others are held fixed. Our experience with functions of several real variables might lead us to suspect that such separate smoothness in not enough.

Exercise 38. *Give an example of a function of two real variables which is C^∞ as a function of x for any fixed value of y and C^∞ as a function of y for any fixed value of x but which fails not in $C^1(\mathbb{R}^2)$.*

We wish to prove that if f is analytic on an (open) polydisc D, then f is represented by a power series. It is convenient to begin this discussion with a definition of the notion of convergence obtained in this theorem.

Definition 58. *Let $\mathcal{A}$ be a countable index set. A series of functions $\sum_{\alpha \in \mathcal{A}} a_\alpha(z)$ converges* **normally** *in Ω if for every compact subset K of Ω, the series $\sum \sup_K |a_\alpha(z)|$ is convergent.*

Exercise 39. *Suppose $\zeta \in \partial_0 D(0, r)$. Find a poser series representing*

$$\frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

in $D(0, r)$ and show that it converges normally.

4.3. **The Inhomogeneous Cauchy-Riemann Equations.** Let $\Omega \subseteq \mathbb{C}^n$, $n > 1$. Suppose we are given a $(0, 1)$ from f and we wish to find a function u such that

$$\bar{\partial} u = f. \tag{20}$$

Since f is a $(0, 1)$ form, $f = \sum_{j=1}^n f_j d\bar{z}_j$. A less succinct way to write (20)

$$\sum_{j=1}^n \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j = \sum_{j=1}^n f_j d\bar{z}_j.$$

Thus given the n functions f_j, we aim to solve the system

$$\frac{\partial u}{\partial \bar{z}_j} = f_j, \quad 1 \leq j \leq n$$

for u.

We pause to recall that we considered this problem briefly already in the case $n = 1$. Reread Theorem 16. A rephrasing the second part is the following:

Theorem 59. *If $f \in C_0^k(\Omega)$ with $k > 1$,*

$$u(z) := \int \frac{f(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta} \tag{21}$$

is in $C^k(\Omega)$ with $\frac{\partial u}{\partial \bar{z}} = f$.

We seek a generalization for $n > 1$. We first note that since $\bar{\partial}^2 = 0$, in order for (20) to have a solution, it is necessary that $\bar{\partial} f = 0$. In other words,

$$\begin{aligned} 0 &= \bar{\partial}(\sum_{j=1}^{n} f_j d\bar{z}_j) \\ &= \sum_{j=1}^{n} \bar{\partial} f_j \wedge d\bar{z}_j \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial f_j}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_j. \end{aligned}$$

If $j = k$, $d\bar{z}_k \wedge d\bar{z}_j = 0$. If $k \neq j$, the above sum involves two terms that may be combined, and when combined, the resulting term must be 0. Thus we require

$$\left(\frac{\partial f_j}{\partial \bar{z}_k} - \frac{\partial f_k}{\partial \bar{z}_j} \right) d\bar{z}_k \wedge d\bar{z}_j = 0 d\bar{z}_k \wedge d\bar{z}_j. \tag{22}$$

We describe this condition on the form f as a **compatibility condition**.

Exercise 40. *Is there a corresponding compatibility condition when $n = 1$? When $n = 2$, give an example of a $(0, 1)$ form f that satisfies the compatibility condition and an example of a $(0, 1)$ form g that does not.*

We may now state the theorem.

Theorem 60 (Hörmander's Theorem 2.3.1)**.** *Suppose $n > 1$. Let $f_j \in C_0^k(\mathbb{C}^n)$, for $1 \leq j \leq n$ and $k \geq 1$ and that (22) is satisfied. Then there is a function $u \in C_0^k(\mathbb{C}^n)$ satisfying (20).*

The theorem as stated is *not* true when $n = 1$; this theorem includes as a conclusion that if the f_j which are the coefficients of the 1-form f are compactly supported, *then the solution function u is compactly supported.* To see that this fails when $n = 1$, consider f smooth with compact support and such that $\int f \, dz \wedge d\bar{z} \neq 0$.

Suppose the solution u had compact support contained in $B(0,R)$. Then

$$\begin{aligned}
0 &= \int_{|z|=R} u(z)\,dz \\
&= \int_{B(0,R)} du \wedge dz \\
&= \int_{B(0,R)} \frac{\partial u}{\partial \bar{z}}\, d\bar{z} \wedge dz \\
&= \int_{B(0,R)} f\, d\bar{z} \wedge d \neq 0.
\end{aligned}$$

This is a contradiction, and thus u can not be compactly supported even if f is if $n = 1$.

We now proceed to prove the several complex variables result. Note that the method of proof is similar to that of Theorem 1.2.2 even though the result differs somewhat.

Proof. For $z = (z_1, \ldots, z_n)$, define

$$\begin{aligned}
u(z) &= \frac{1}{2\pi i} \int \frac{f_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1}\, d\zeta \wedge d\bar{\zeta} \\
&= \frac{1}{2\pi i} \int \frac{f_1(w + z_1, z_2, \ldots, z_n)}{w}\, dw \wedge d\bar{w}.
\end{aligned}$$

Since f_1 and its first k derivatives are continuous with compact support, and since $1/w$ is locally integrable on $\mathbb{C}$, it is legitimate to obtain derivatives of u up to order k by differentiating under the integral sign, and these derivatives are continuous. Thus certainly $u \in C^k(\mathbb{C}^n)$. It also follows from Theorem 16 that $\partial u/\partial \bar{z}_1 = f_1$.

Now consider $j > 1$. Since $k \geq 1$, all derivatives $\partial u/\partial \bar{z}_j$ are also obtained by differentiating under the integral sign. Thus

$$\begin{aligned}
\frac{\partial u}{\partial \bar{z}_j} &= \frac{1}{2\pi i} \int \frac{1}{w} \frac{\partial f_1}{\partial \bar{z}_j}(w + z_1, z_2, \ldots, z_n)\, dw \wedge d\bar{w} \\
&= \frac{1}{2\pi i} \int \frac{1}{w} \frac{\partial f_j}{\partial \bar{z}_1}(w + z_1, z_2, \ldots, z_n)\, dw \wedge d\bar{w} \\
&= \frac{1}{2\pi i} \int \frac{1}{\zeta - z_1} \frac{\partial f_j}{\partial \bar{\zeta}}(\zeta, z_2, \ldots, z_n)\, d\zeta \wedge d\bar{\zeta} \\
&= f_j(z_1, \ldots, z_n),
\end{aligned}$$

where we have used both the compatibility condition and the Cauchy Integral Formula on a disc in $\mathbb{C}$ large enough so that $f_j(\zeta, z_2, \ldots, z_n) = 0$ on its boundary.

It remains to prove that u has compact support. Indeed, if we go back to the definition of u, since f_1 has compact support contained in some $\overline{D(0,R)} := \{\, z : |z_j| \leq R, 1 \leq j \leq n \,\}$, regardless of the value of z_1, $u(z_1, z_2, \ldots, z_n) = 0$ in $S := \{\, |z_j| > R : 2 \leq j \leq n \,\}$. S^c is closed and contains the support of u, but S^c is not itself compact. However, we also know from Theorem 16 that u is analytic off the support of f_1, i.e., on $\overline{D(0,R)}^c$. Since this latter subset is connected and since u is analytic and identically zero on the open subset S, By analytic continuation, u is identically zero on $\overline{D(0,R)}^c$. This proves that u has compact support. and completes the proof of the theorem. □

It is perhaps initially surprising that our first application of this theorem on the solution of the *inhomogeneous* Cauchy-Riemann equations is to a problem of *holomorphic* extension. We ask a basic question: Do there exist open sets Ω in $\mathbb{C}^n$ for which there is an open set Ω' properly containing Ω having the property that for every $u \in A(\Omega)$ there exists $U \in A(\Omega)$ with $U|_\Omega = u$?

If this question has never occurred to you, it is perhaps because when $n = 1$, such holomorphic extension is never possible.

Set

$$D(0, r) = \{ (z_1, z_2) \,|\, |z_1|, |z_2| < r \}.$$

Let

$$\Omega = D(0, 1) \setminus \overline{D(0, 1/2)}.$$

We claim that every holomorphic function on Ω extends holomorphically to $D(0, 1)$. We can show this with one-variable techniques.

Suppose u is a holomorphic function on Ω. Define a new function on $D(0, 3/4)$ by

$$U(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta|=\frac{3}{4}} \frac{u(\zeta, z_2)}{\zeta - z_1} \, d\zeta.$$

By differentiating under the integral sign, one sees that U is holomorphic in $D(0, 3/4)$. Furthermore, for z_2 satisfying $\frac{1}{2} < |z_2| < \frac{3}{4}$, by the Cauchy integral formula, the given integral is also equal to $u(z_1, z_2)$. Since u and U are both holomorphic on $D(0, 3/4) \setminus \overline{D(0, 1/2)}$ and agree on $\{ |z_1| < 3/4, \ 1/2 < |z_2| < 3/4 \}$, and since $D(0, 3/4)$ is connected, by the uniqueness of analytic continuation, they agree on the entire set. We have thus achieved the desired extension of h.

Theorem 61 (Hartogs)**.**

Exercise 41. *Show that, if $\Omega \subset \mathbb{C}^n$ is convex, for any open set Ω' properly containing Ω, there is an element of $A(\Omega)$ that does not extend to an element of $A(\Omega')$.*

Exercise 42. *Let f be holomorphic on $\Omega \subseteq \mathbb{C}^n$, $n > 1$. Fix $\alpha \in \mathbb{C}$. If the level set $S := \{ z \in \Omega : f(z) = \alpha \}$ is not empty, show what S is not contained in a compact subset of Ω. Compare and contrast this with the situation when $n = 1$.*

Exercise 43. *Formulate and prove a version of the maximum principle for holomorphic functions of several complex variables.*

Perhaps even more surprising is the fact that it is sometimes possible to extend functions satisfying an appropriate condition on the boundary of a domain to analytic functions inside. Before we state the theorem, we make some definitions and try to motivate some of the hypotheses that appear in the theorem.

Definition 62. *Let Ω be an open subset of $\mathbb{C}^n$ with boundary $\partial\Omega$. A real-valued function ρ on $\mathbb{C}^n$ is a* **defining function** *for $\partial\Omega$ if $\rho(z) = 0$ if and only if $z \in \partial\Omega$ and $\nabla\rho(z) \neq 0$ for $z \in \partial\Omega$.*

Example 63. $\rho(z) = \sum_{j=1}^n |z_j|^2 - 1$ *is a defining function for the boundary of the unit ball $B(0, 1)$, but ρ^2 is not. $\rho(z, w) = \mathrm{Im}(w)$ is a defining function for the boundary of the half-space $\{ (z = x + iy, w = u + iv) \in \mathbb{C}^2 : v > 0 \}$.*

We will need the following proposition:

Proposition 64. *Let ρ be a C^1 defining function for $\partial\Omega$ and suppose g is a C^1 function that vanishes on $\partial\Omega$. Then there exists h continuous such that $g = h\rho$.*

Proof. We give a sketch. We work locally. By translating if necessary, we may assume $0 \in \partial\Omega$. In a small neighborhood V of the origin we may make a change of variables so that $\partial\Omega \cap V$ has defining equation $y_n = \operatorname{Im}(z_n) = 0$. Suppose g vanishes on $\partial\Omega$. We use the notation $z' := (z_1, \ldots, z_{n-1})$ and write $z_n := x_n + iy_n$. Then in V

$$\begin{aligned} g(z', z_n) &= g(z', x_n + iy_n) - g(z', x_n + i0) \\ &= \int_0^{y_n} \frac{\partial g}{\partial y_n}(z', x_n + it)\, dt \\ &= \int_0^1 \frac{\partial g}{\partial y_n}(z', x_n + iy_n s)\, ds \cdot y_n \\ &= h(z)\rho(z). \end{aligned}$$

□

With this set-up, we can begin discussing the result we aim to prove. Suppose u is smooth on a neighborhood of $\overline{\Omega}$. Can we identify a condition on u which is necessary if there is to exists U smooth *and analytic* on a neighborhood of $\overline{\Omega}$ with $U|_{\partial\Omega} = u|_{\partial\Omega}$?

If such a U exists, $U - u$ vanishes on $\partial\Omega$ and so there exists h such that $U - u = h\rho$. Apply $\bar\partial$. $\bar\partial U - \bar\partial u = \rho\bar\partial h - h\bar\partial rho$. Since U is analytic, $\bar\partial U = 0$. Also, *for all $z \in \partial\Omega$,* $\rho(z) = 0$. Therefore

$$-\bar\partial u(z) = h(z)\bar\partial\rho(z), \quad z \in \partial\Omega. \tag{23}$$

We can rewrite this condition on u in two different ways. First, (23) says that at every point $z \in \partial\Omega$, the form $\bar\partial u(z)$ is a scalar multiple of $\bar\partial\rho(z)$. Thus $\bar\partial u(z) \wedge \bar\partial\rho(z) = 0$ on $\partial\Omega$. Another way to express this is to say that if a is orthogonal to $\bar\partial\rho(z)$, it is also orthogonal to $\bar\partial u(z)$, i.e.,

$$\sum_{j=1}^n a_j \frac{\partial u}{\partial \bar z_j}(z) = 0 \quad \text{whenever} \quad \sum_{j=1}^n a_j \frac{\partial \rho}{\partial \bar z_j}(z) = 0. \tag{24}$$

We call these the *tangential Cauchy-Riemann equations* - we are only asking that u be annihilated by an operator $\sum a_j \partial/\partial\bar z_j$ for certain a tangent to the boundary identified by the condition $\sum a_j \partial\rho/\partial\bar z_j$.

Is it true that whenever u satisfies (24) on $\partial\Omega$ then it is the restriction of some U analytic on an open set containing $\overline{\Omega}$? The answer is no.

Example 65. *Return to $\Omega = \{\,(z,w) : \operatorname{Im} w > 0\,\}$. The defining equation for $\partial\Omega$ is $\rho(z,w) = \frac{w - \bar w}{2i}$. Since*

$$0 = a_1 \frac{\partial\rho}{\partial\bar z} + a_2 \frac{\partial\rho}{\partial\bar w} = a_2(-\frac{1}{2i}) \quad \textit{iff} \quad a_2 = 0,$$

the condition imposed on u by (24) is $\frac{\partial u}{\partial \bar z} = 0$ on $\partial\Omega$.

For this boundary, however, even if we relax our requirements and simply try to find some small open subset V of Ω with the property that every u satisfying (24) extends to a function analytic on V, we will be unsuccessful. For example, take $(z_0, w_0) \in V$ (so that $\operatorname{Im} w_0 > 0$). Consider $u(z,w) := (w - w_0)^{-1}$. This is smooth *on* $\partial\Omega$ *since* $\operatorname{Im} w = 0$ *there and* $\partial u/\partial\bar{z} = 0$*, but* u *does not agree with an analytic function of* (z, w) *on* V*.*

We therefore need additional hypotheses on u and/or additional hypotheses on Ω. A full exploration of this question is well beyond the scope of this course. The interested reader should see the sections concerning holomorphic extension of CR functions in Boggess's book [Bog91]. For now we consider the theorem presented by Hörmander which addresses one holomorphic extension situation.

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