Computer Lab #5: MATH 471
Wednesday, October 8, 2005.

Again, we consider the problem of computing wavefront errors that is a requirement of adaptive optics systems. The differential equation that must be solved, has the form

$$\nabla^2 u(x,y) = \nabla \cdot g(x,y), \quad 0 < x, y < 1, \tag{1}$$

where $\nabla^2 = \partial/\partial x^2 + \partial/\partial y^2$ is the Laplacian operator, $\nabla\cdot$ divergence operator, and $u = 0$ on the boundary of $\Omega = [0,1] \times [0,1]$. Discretization of (1) yields a linear system of the form

$$\mathbf{Ax} = \mathbf{b}, \tag{2}$$

where $A \in \mathbb{R}^{N\times N}$ is the discrete version of the ∇^2 operator; $\mathbf{b} \in \mathbb{R}^N$ is the discretization of $\nabla \cdot g$, and $\mathbf{x} \in \mathbb{R}^N$ is the unknown solution.

Previously, we solved this system using Gaussian elimination, the Cholesky factorization, and a number of iterative methods, such as Gauss-Seidel, Jacobi, and SOR. This time we seek approximate solutions via the conjugate gradient iteration.

PROBLEM:

1. Modify `AOTwoD.m` so that it uses the CG iteration to approximately solve (2).

2. **Problem from the homework:** Solve the linear system created by `FD_2D.m` with `cg.m` using the preconditioner given by the Q-matrix used in the SSOR algorithm, i.e.

$$Q(\omega) = (\omega(2-\omega))^{-1}(D - \omega L)D^{-1}(D - \omega U), \tag{3}$$

   where $A = D - L - U$ as usual. Note that Q is symmetric, which is a requirement for any preconditioning matrix. Create a single figure which contains residual plots for the values of $0 < \omega < 2$ given by $\omega = .25k$ for $k = 1, \ldots, 7$. What of these values of ω results in the optimal preconditioner for this problem?

3. Let $Q(\omega) = C(\omega)^T C(\omega)$ be given in (3). When preconditioning is used, the new coefficient matrix, then, is $C(\omega)^{-T} A C(\omega)^{-1}$. Create a single figure that contains plots of the eigenvalues A and of $C(\omega)^{-T} A C(\omega)^{-1}$ for two values of ω including the optimal value computed in part (a). To do this, you'll need to use the commands

```
>> spec = eig(full(C'\ A / C));
>> spec = sort(real(spec));
```

   Then plot the results in a single figure, and answer the following questions: Which matrix has the smallest ℓ_2 condition number $\kappa(C(\omega)^{-T} A C(\omega)^T)$? Which matrix has the "most clustered" eigenvalues?