ASSIGNMENT #4: MATH 471
Due: Wednesday, October 8, 2008.

1. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   (a) Compute the SVD of $\mathbf{A}$ by hand. Use MATLAB **only** to check your work.

   (b) Using your result in (a), compute the psuedo-inverse of $\mathbf{A}$.

   (c) Using your answer in (b), compute the least squares solution of $\mathbf{Ax} = \mathbf{b}$ with minimum norm. Is it a solution of $\mathbf{Ax} = \mathbf{b}$?

   (d) The *normal equations* of the least squares problem $\left\{\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|^2\right\}$ have the form

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}.$$

   If $\mathbf{A}^T\mathbf{A}$ is invertible, then the least squares solution is unique and has the form $\mathbf{x}_{\text{LS}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$, in which case, $\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$, where "$\dagger$" denotes psuedo-inverse. Using your computation of $\mathbf{A}^\dagger$ in (b), show that this is true for $\mathbf{A}$ defined above.

2. Suppose that if $\mathbf{A} = \mathbf{VDV}^T$, where $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_n)$ is $n \times n$ diagonal, and $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$ with $\mathbf{v}_i$ an $n \times 1$ vector for all i, and $\mathbf{V}^T\mathbf{V} = \mathbf{I}$ with $\mathbf{I}$ the $n \times n$ identity matrix.

   (a) Show that the eigenvalue/vector pairs for $\mathbf{A}$ are $\{(\lambda_i, \mathbf{v}_i)\}_{i=1}^n$, i.e. that

$$\mathbf{Av}_i = \lambda_i \mathbf{v}_i, \quad i = 1, \dots, n.$$

   (b) Show that if $\mathbf{A}$ is positive definite, $\lambda_i > 0$ for $i = 1, \dots, n$.

   (c) Show that $\mathbf{A}^k = \mathbf{VD}^k\mathbf{V}^T$. What does this say about the eigenvalue/vector pairs for $\mathbf{A}^k$?

3. In lab, we implemented the TSVD filter to obtain a reconstructed images in `TSVDOneD.m` and `TSVDTwoD.m`. If the model is $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} = \mathbf{USV}^T$ the SVD of an $n \times n$ matrix $\mathbf{A}$, this approach can be expressed as follows:

$$\sum_{i=1}^{n} \phi_i^\alpha \frac{\mathbf{u}_i^T\mathbf{b}}{\sigma_i}\mathbf{v}_i, \quad \phi_i^\alpha = \begin{cases} 1 & \sigma_i > \alpha \\ 0 & \sigma_i \le \alpha. \end{cases}$$

   The Tikhonov filter is expressed instead as

$$\phi_i^\alpha = \frac{\sigma_i^2}{\alpha + \sigma_i^2}, \quad i = 1, \dots, n.$$

   Modify `TSVDOneD.m` and `TSVDTwoD.m` so that the Tikhonov filter is implemented instead of the TSVD filter. Which filter seems to work better? Hand in a listing of your code.