9.5: 4, 6, 10a)

4. $f(x)=xe^{-x},\ x\ge 0,\quad f(x)=0,\ x<0$

a) $f(x)\ge 0\ \forall x,$

$$\int_0^\infty xe^{-x} = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx \qquad u=x \quad dv=e^{-x}dx$$

$$= \lim_{t\to\infty}\left\{-xe^{-x}\Big|_0^t - \int_0^t e^{-x}\,dx\right\}$$

$$= \lim_{t\to\infty}\left\{te^{-t} - e^{-x}\Big|_0^t\right\}$$

$$= \lim_{t\to\infty}\left\{-te^{-t} - e^{-t} + 1\right\}$$

$$= 0 - 0 + 1 = 1.$$

b) $P(1 \le X \le 2) = \int_1^2 xe^{-x}\,dx$

$$= -xe^{-x}\Big|_1^2 - e^{-x}\Big|_1^2$$

$$= -2e^{-2} + e^{-1} - e^{-2} + e^{-1}$$

$$= -3e^{-2} + 2e^{-1}$$

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6. $f(x) = kx^2(1-x) = k(x^2 - x^3)$

a) $0 \le x \le 1,\ f(x)=0,\ x<0,\ x>1.$

$$\int_0^1 kx^2 - kx^3\,dx = k\frac{x^3}{3} - k\frac{x^4}{4}\Big|_0^1$$

$$= \frac{k}{3} - \frac{k}{4} = \frac{k}{12}$$

So if $k=12$, f is a p.d.f.

b) $P(X \ge 1/2) = 12\int_{1/2}^1 (x^2 - x^3)\,dx$

$$= 12\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_{1/2}^1$$

$$= 12\left(\frac{1}{3} - \frac{1}{4} - \left(\frac{1}{24} - \frac{1}{64}\right)\right)$$

c) $\mu = \int_0^1 x(12x^2 - 12x^3)\,dx$

$$= 12\int_0^1 x^3 - x^4\,dx$$

$$= 12\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1$$

$$= 12\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= 12\left(\frac{1}{20}\right) = \frac{3}{5}.$$

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10 a) $f(x) = \frac{1}{1000}e^{-x/1000}$

i) $P(0 \le X \le 200) = \int_0^{200} f(x)\,dx$

$$= \frac{1}{1000}\cdot(-1000)\,e^{-x/1000}\Big|_0^{200}$$

$$= -e^{-1/5} + 1 = 0.1813$$

ii) $P(X \ge 800) = \int_{800}^\infty f(x)\,dx$

$$= \frac{1}{1000}(-1000)\left(\lim_{t\to\infty} e^{-x/1000}\Big|_{800}^t\right)$$

$$= -\lim_{t\to\infty}\left(e^{-t/1000} - e^{-800/1000}\right)$$

$$= e^{-8/10} = 0.4493$$