HW 8.4 : 19, 24

19. $\int \frac{1}{(x+5)^2(x-1)}\,dx$

$$\frac{1}{(x+5)^2(x-1)} = \frac{A_1}{x+5} + \frac{A_2}{(x+5)^2} + \frac{A_3}{(x-1)}$$

$$\Rightarrow \quad 1 = A_1(x+5)(x-1) + A_2(x-1) + A_3(x+5)^2$$

$\underline{x=1}$: $\quad 1 = 36A_3 \Rightarrow \boxed{A_3 = 1/36}$

$\underline{x=-5}$: $\quad 1 = -6A_2 \Rightarrow \boxed{A_2 = -1/6}$

What about A_1? expand things out

$$1 = A_1(x^2+4x-5) + A_2(x-1) + A_3(x^2+10x+25)$$

$$= \underbrace{\boxed{(A_1+A_3)}}_{=0}x^2 + \overbrace{\boxed{(4A_1+A_2+10A_3)}}^{=0}x + \boxed{(-5A_1-A_2+25A_3)}\ {=1}$$

$$\Rightarrow \quad \boxed{A_1 = -A_3 = -\frac{1}{36}}$$

$$\int \frac{1}{(x+5)^2(x-1)}\,dx = \int \left(-\frac{1}{36}\cdot\frac{1}{x+5} - \frac{1}{6}\cdot\frac{1}{(x+5)^2} + \frac{1}{36}\cdot\frac{1}{x-1}\right)dx$$

$$= -\frac{1}{36}\ln|x+5| + \frac{1}{6}(x+5)^{-1} + \frac{1}{36}\ln|x-1| + C.$$

24. $\int \frac{x^2-x+6}{x^3+3x}\,dx$

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A_1}{x} + \frac{A_2x+A_3}{x^2+3}$$

$$\Rightarrow \quad x^2-x+6 = A(x^2+3) + (A_2x+A_3)x$$

$\underline{x=0}$: $\quad 6 = 3A_1 \Rightarrow \boxed{A_1 = 2}$

otherwise, expand

$$x^2-x+6 = (A_1+A_2)x^2 + (3A_1+A_3)x + 3A_1$$

$$3A_1 = 6 \Rightarrow \boxed{A_1 = 2}$$

$$3A_1 + A_3 = -1 \Rightarrow A_3 = -1-6 = \boxed{-7 = A_3}$$

$$A_1 + A_2 = 1 \Rightarrow A_2 = 1-2 = \boxed{-1 = A_2}$$

So

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int\left(\frac{2}{x} + \frac{-x + -7}{x^2+3}\right)dx$$

$$= 2\ln|x| - \int \frac{x}{x^2+3}\,dx - 7\int\frac{dx}{x^2+3}$$

$u = x^2+3$

$$= 2\ln|x| - \frac{1}{2}\int\frac{du}{u} - 7\left(\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right) + C$$

$$= 2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{7}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$